SOLUTIONS to Exercises for Physics for Physiotherapy Technology

203-9P1-DW Fall 2022

by Chris Roderick, Ph.D.

Created 2019 August 16th.
Version α13, TEXify-ed November 21, 2022.
Continuous-update version
Fall 2022 α13

Why practice?

Learning is about changing how you think. The most effective way to do that is to start by knowing how you think *now*. So always begin solving a problem by writing out what you think the answer will be like. Yes!, make a guess! But after that, do procede using the appropriate methods, and ***don't skip steps***. At the end, compare your result with your initial guess. Are they different? If yes, how do they differ? Are they different by a few percent? Or are they completely opposite?

That will be the place to pause and reflect on how you are thinking about these situations, and figure out what you need to change in your thinking. I'm here to help with that step, but it will go much faster if you contribute towards identifying where you need the help. Doing the exercises is the place where you work on that analysis.

When practicing problems spend the majority of your time being very explicit about the context, assumptions, and methods that you will be using in your process – that is, write everything out. This does take time, but it practices what is important: reasoning about the physics. You might not get as many problems done, but you will have done them better and gained more.

So now, let's get to work.

Contents

Chapter 0

Preparation

0.1 Units

0.1.1 Lengths

Exr 0.1.01 1 m = 100 cm = 1 000 mm
Exr 0.1.02 1 cm = 0.01 m = 10 mm
Exr 0.1.03 3.205 m = 320.5 cm = 3 205 mm
Exr 0.1.04 0.829 m = 82.9 cm = 829 mm
Exr 0.1.05 7 cm = 0.07 m = 70 mm
Exr 0.1.06 237 cm = 2.37 m
Exr 0.1.07 15 mm = 1.5 cm = 0.015 m
Exr 0.1.08 29.45 cm = 294.5 mm = 0.2945 m
Exr 0.1.09 14 in = 35.56 cm
Exr 0.1.10 42 in = 3.5 ft = 1.067 m
Exr 0.1.11 88 in = 2.235 m
Exr 0.1.12 1 ft = 30.48 cm
Exr 0.1.13 5’6” = 5 ft 6 in = 1.68 m
Exr 0.1.14 25 cm = 9.8 in
Exr 0.1.15 1.00 m = 3 ft and 3.37 in
Exr 0.1.16 1.956 m = 6 ft and 5 in

0.1.2 Areas & Volumes

Exr 0.1.17 1 m^2 = 10 000 cm^2
Exr 0.1.18 400 cm^2 = 0.0400 m^2
Exr 0.1.19 0.0820 m^2 = 820 cm^2
Exr 0.1.20 37 cm^2 = 3 700 mm^2
Exr 0.1.21 1 in^2 = 6.45 cm^2 = 645 mm^2
Exr 0.1.22 1 ft^2 = 144 in^2 = 929 cm^2
Exr 0.1.23 200 ft^2 = 18.58 m^2
Exr 0.1.24 50 cm^2 = 7.75 in^2
Exr 0.1.25 7 L = 7 000 cm^3 = 7 000 mL
Exr 0.1.26 375 mL = 0.375 L
Exr 0.1.27 0.520 L = 520 mL
Exr 0.1.28 1 m^3 = 1 000 000 cm^3 = 1 000 L
Exr 0.1.29 1 in^3 = 16.39 cm^3
Exr 0.1.30 1 ft^3 = 28 317 cm^3 = 28.3 L
Exr 0.1.31 10 L = 0.353 ft^3
Exr 0.1.32 625 mL = 38.1 in^3

0.1.3 Time

When working with measurements of time remember that (in this context) the non-standard symbols “d” for days, “h” for hours, and “min” for minutes are used. These are not to be confused with the Metric prefixes “d” for “deci” (10^{-1}), and “h” for “hecto” (10^{+2}). Be aware of the context.

For these conversions recall the definitions: $1\,\text{d} = 24\,\text{h}$, $1\,\text{h} = 60\,\text{min}$, and $1\,\text{min} = 60\,\text{s}$.

Exr 0.1.33 90 s = 1.5 min
Exr 0.1.34 250 s = 4.17 min
Exr 0.1.35 1 000 s = 16.67 min
Exr 0.1.36 15 s = 0.25 min
Exr 0.1.37 10.0 min = 600 s = 0.167 h
Exr 0.1.38 411 min = 24.7×10^3 s = 6.85 h
Exr 0.1.39 900 min = 54.0×10^3 s = 15.0 h
Exr 0.1.40 $\frac{1}{5}$ min = 12 s = $\frac{1}{300}$ h
Exr 0.1.41 0.20 h = 12 min = 8.3×10^{-3} d
Exr 0.1.42 1.5 h = 90 min = 0.063 d
Exr 0.1.43 8 h = 480 min = $\frac{1}{3}$ d
Exr 0.1.44 30 h = 1 800 min = 1.25 d
Exr 0.1.45 100 h = 6 000 min = 4.17 d
Exr 0.1.46 888 h = 53.3×10^3 min = 37 d
Exr 0.1.47 $\frac{1}{10}$ d = 2.4 h = 144 min
Exr 0.1.48 $\frac{1}{3}$ d = 8 h = 480 min
Exr 0.1.49 7 d = 168 h = 0.6×10^6 s
Exr 0.1.50 30 d = 720 h = 2.6×10^6 s
Exr 0.1.51 365 d = 8 760 h = 31.5×10^6 s

0.1.4 Rates

Exr 0.1.52 1 m/s = 60 m/min = 3 600 m/h
Exr 0.1.53 1.28 m/s = 77 m/min = 4 620 m/h
Exr 0.1.54 2.5 m/s = 150 m/min = 9 000 m/h
Exr 0.1.55 90 km/h = 25 m/s
Exr 0.1.56 100 km/h = 27.8 m/s
Exr 0.1.57 10 m/s = 36 km/h
Exr 0.1.58 343 m/s = 1 235 km/h
Exr 0.1.59 5.0 ft/s = 5.5 km/h
Exr 0.1.60 50 mL/s = 3.0 L/min
Exr 0.1.61 8.33 L/s = 0.500 m^3/min
Exr 0.1.62 2 L/min = 33.3 mL/s
Exr 0.1.63 0.370 m^3/min = 6.17 L/s

0.1.5 Masses & Forces

Exr 0.1.64 3.2 kg = 3 200 g
Exr 0.1.65 487 g = 0.487 kg
Exr 0.1.66 13 g = 0.013 kg
Exr 0.1.67 18 lb = 8.2 kg
Exr 0.1.68 145 lb = 65.8 kg
Exr 0.1.69 80 kg = 176 lb
Exr 0.1.70 72 kg weighs 706 N
Exr 0.1.71 8.56 kg weighs 84.0 N
Exr 0.1.72 150 lb weighs 667 N
Exr 0.1.73 169 lb weighs 750 N

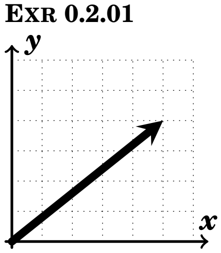
0.2 Vectors

0.2.1 Components, Magnitudes & Angles

From Components to Magnitudes & Angles

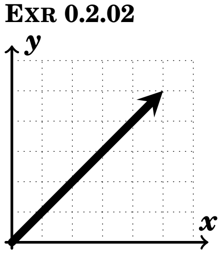
For each of the vectors below find its components, then calculate its magnitude and the angle it makes with the $+x$-axis. (The angle measured counter-clockwise is positive.) In these exercises the grid size is 1 cm.

Exr 0.2.01



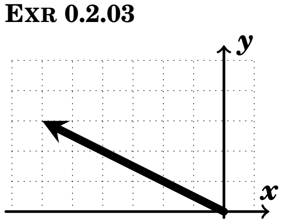
$$A_x = +5\,\mathrm{cm} \tag{1}$$
$$A_y = +4\,\mathrm{cm} \tag{2}$$
$$A = 6.403\,\mathrm{cm} \tag{3}$$
$$\theta = 38.7^\circ \tag{4}$$

Exr 0.2.02



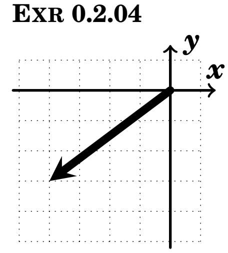
$$A_x = +5\,\mathrm{cm} \tag{5}$$
$$A_y = +5\,\mathrm{cm} \tag{6}$$
$$A = 7.071\,\mathrm{cm} \tag{7}$$
$$\theta = 45.0^\circ \tag{8}$$

Exr 0.2.03



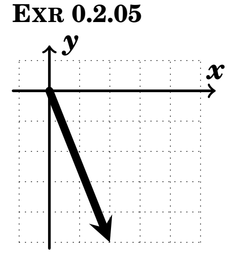
$$A_x = -6\,\mathrm{cm} \tag{9}$$
$$A_y = +3\,\mathrm{cm} \tag{10}$$
$$A = 6.708\,\mathrm{cm} \tag{11}$$
$$\theta = 153.4^\circ \tag{12}$$

Exr 0.2.04



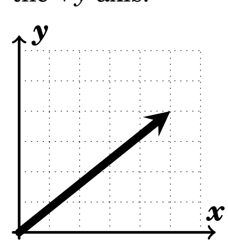
$$A_x = -4\,\mathrm{cm} \tag{13}$$
$$A_y = -3\,\mathrm{cm} \tag{14}$$
$$A = 5.000\,\mathrm{cm} \tag{15}$$
$$\theta = -143.1^\circ \text{ or } 216.9^\circ \tag{16}$$

Exr 0.2.05



$$A_x = +2\,\mathrm{cm} \tag{17}$$
$$A_y = -5\,\mathrm{cm} \tag{18}$$
$$A = 5.385\,\mathrm{cm} \tag{19}$$
$$\theta = -68.2^\circ \text{ or } 291.8^\circ \tag{20}$$

Exr 0.2.06 Find the angle between this vector and the $+y$-axis:

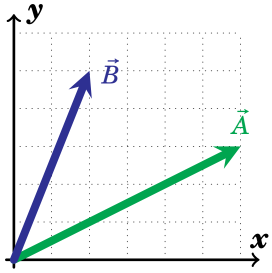


$$A_x = +5\,\mathrm{cm} \tag{21}$$
$$A_y = +4\,\mathrm{cm} \tag{22}$$
$$A = 6.403\,\mathrm{cm} \tag{23}$$
$$\theta = 51.3^\circ \tag{24}$$

Angles Between Vectors

For each of the pairs of vectors below find the angle between them.

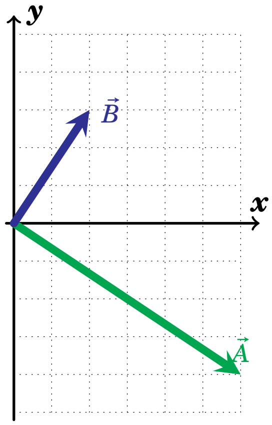
Exr 0.2.07



$$A_x = +6\,\text{cm} \qquad B_x = +2\,\text{cm} \tag{25}$$
$$A_y = +3\,\text{cm} \qquad B_y = +5\,\text{cm} \tag{26}$$
$$\theta_A = 26.6^\circ \qquad \theta_B = 68.2^\circ \tag{27}$$

The angle between the vectors is $\theta_B - \theta_A = 41.6^\circ$.

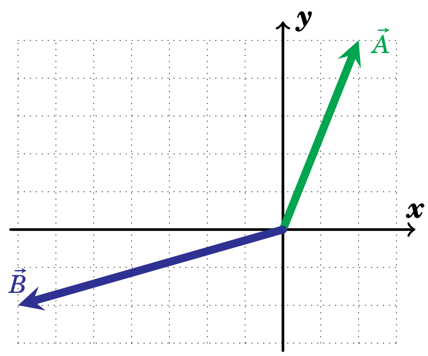
Exr 0.2.08



$$A_x = +6\,\text{cm} \qquad B_x = +2\,\text{cm} \tag{28}$$
$$A_y = -4\,\text{cm} \qquad B_y = +3\,\text{cm} \tag{29}$$
$$\theta_A = -33.7^\circ \qquad \theta_B = 56.3^\circ \tag{30}$$

The angle between the vectors is $\theta_B - \theta_A = 90.0^\circ$.

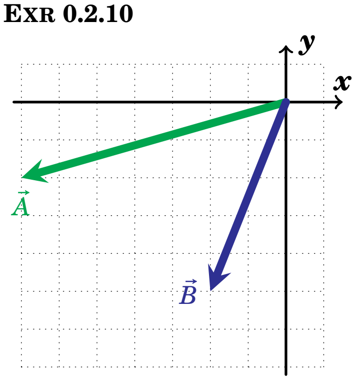
Exr 0.2.09



$$A_x = +2\,\text{cm} \qquad B_x = -7\,\text{cm} \tag{31}$$
$$A_y = +5\,\text{cm} \qquad B_y = -2\,\text{cm} \tag{32}$$
$$\theta_A = 68.2^\circ \qquad \theta_B = 195.9^\circ \tag{33}$$

The angle between the vectors is $\theta_B - \theta_A = 127.7^\circ$.

Exr 0.2.10



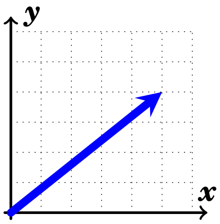
$$A_x = -7\,\text{cm} \qquad B_x = -2\,\text{cm} \tag{34}$$
$$A_y = -2\,\text{cm} \qquad B_y = -5\,\text{cm} \tag{35}$$
$$\theta_A = 195.9^\circ \qquad \theta_B = 248.2^\circ \tag{36}$$

The angle between the vectors is $\theta_B - \theta_A = 52.3^\circ$.

From Magnitudes & Angles to Components

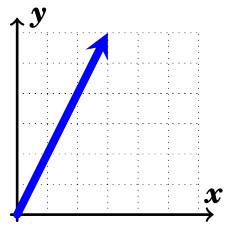
For each of the vectors below find its components, then sketch it on the provided grid. (In these exercises the grid size is 1 cm.)

Exr 0.2.11 Draw the vector of magnitude 6.403 cm directed 38.7° counter-clockwise from the $+x$-axis.



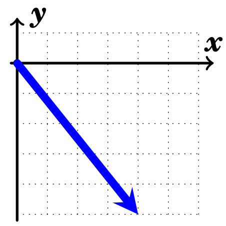
$$A_x = 6.403\,\text{cm}\cos 38.7^\circ = +5\,\text{cm}$$
$$A_y = 6.403\,\text{cm}\sin 38.7^\circ = +4\,\text{cm}$$

Exr 0.2.12 Draw the vector of magnitude 6.708 cm directed 63.4° counter-clockwise from the $+x$-axis.



$$A_x = 6.708\,\text{cm}\cos 63.4^\circ = +3\,\text{cm}$$
$$A_y = 6.708\,\text{cm}\sin 63.4^\circ = +6\,\text{cm}$$

Exr 0.2.13 Draw the vector of magnitude 6.403 cm directed 51.3° clockwise from the $+x$-axis.



"Clockwise from the $+x$-axis" means below the $+x$-axis.

$$A_x = 6.403\,\text{cm}\cos(-51.3^\circ) = +4\,\text{cm}$$
$$A_y = 6.403\,\text{cm}\sin(-51.3^\circ) = -5\,\text{cm}$$

0.2.2 Sums of Vectors

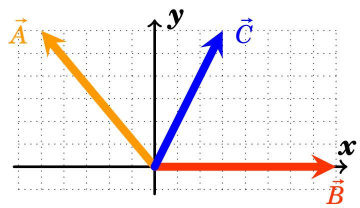
In this set of exercises we will practice the summation of vectors. In each exercise you will need to find the components of the vectors involved. Be explicit and careful with the units of the quantities you are using and calculating.

Sum of Pairs of vectors

In the exercises below each vector is a position, and each grid square corresponds to one centimetre (1 cm) of distance.

For each of the pairs of vectors below calculate their sum. Find its components, its magnitude, and the angle it makes with the $+x$-axis. (The give vectors can be called $\vec{A}$ and $\vec{B}$. Call the sum $\vec{C} = \vec{A} + \vec{B}$.)

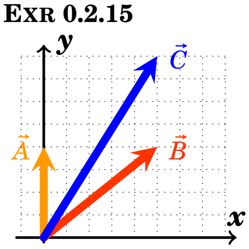
Exr 0.2.14



$$A_x = -5\text{ cm} \qquad B_x = +8\text{ cm} \qquad C_x = +3\text{ cm} \tag{37}$$
$$A_y = +6\text{ cm} \qquad B_y = 0\text{ cm} \qquad C_y = +6\text{ cm} \tag{38}$$

The magnitude and direction of the vector $\vec{C}$ are $C = 6.71$ cm and $\theta_C = 63.4°$.

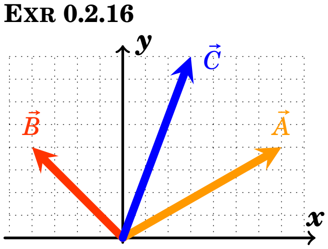
Exr 0.2.15



$$A_x = 0\text{ cm} \qquad B_x = +5\text{ cm} \qquad C_x = +5\text{ cm} \tag{39}$$
$$A_y = +4\text{ cm} \qquad B_y = +4\text{ cm} \qquad C_y = +8\text{ cm} \tag{40}$$

The magnitude and direction of the vector $\vec{C}$ are $C = 9.43$ cm and $\theta_C = 58.0°$.

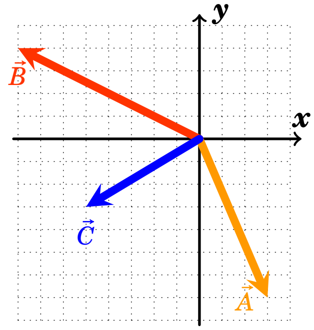
Exr 0.2.16



$$A_x = +7\text{ cm} \qquad B_x = -4\text{ cm} \qquad C_x = +3\text{ cm} \tag{41}$$
$$A_y = +4\text{ cm} \qquad B_y = +4\text{ cm} \qquad C_y = +8\text{ cm} \tag{42}$$

The magnitude and direction of the vector $\vec{C}$ are $C = 8.54$ cm and $\theta_C = 69.4°$.

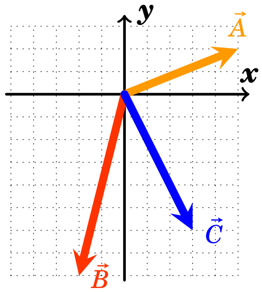
Exr 0.2.17



$$
\begin{array}{llll}
A_x = +3\,\text{cm} & B_x = -8\,\text{cm} & C_x = -5\,\text{cm} & (43)\\
A_y = -7\,\text{cm} & B_y = +4\,\text{cm} & C_y = -3\,\text{cm} & (44)
\end{array}
$$

The magnitude and direction of the vector $\vec{C}$ are $C = 5.83\,\text{cm}$ and $\theta_C = -149°$ (or $\theta_C = 211°$).

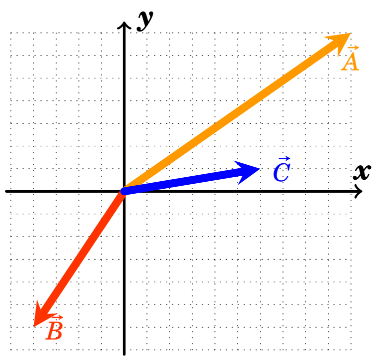
Exr 0.2.18



$$
\begin{array}{llll}
A_x = +5\,\text{cm} & B_x = -2\,\text{cm} & C_x = +3\,\text{cm} & (45)\\
A_y = +2\,\text{cm} & B_y = -8\,\text{cm} & C_y = -6\,\text{cm} & (46)
\end{array}
$$

The magnitude and direction of the vector $\vec{C}$ are $C = 6.71\,\text{cm}$ and $\theta_C = -63.4°$ (or $\theta_C = 297°$).

Exr 0.2.19



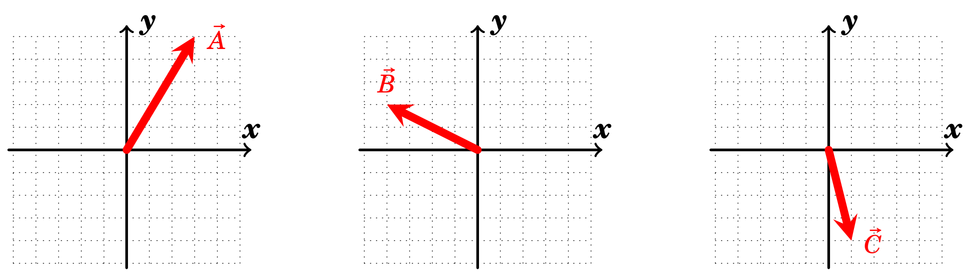
$$
\begin{array}{llll}
A_x = +10\,\text{cm} & B_x = -4\,\text{cm} & C_x = +6\,\text{cm} & (47)\\
A_y = +7\,\text{cm} & B_y = -6\,\text{cm} & C_y = +1\,\text{cm} & (48)
\end{array}
$$

The magnitude and direction of the vector $\vec{C}$ are $C = 6.08\,\text{cm}$ and $\theta_C = 9.46°$.

Comparing Sums

In the exercises below each vector is a force, and each grid square corresponds to one newton (1 N) of force. For the sets of vectors below find the pair of vectors whose sum (the resultant) has the greatest magnitude.

Exr 0.2.20



With three vectors, there are three possible pairs.

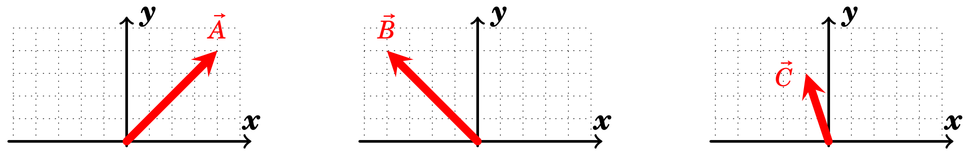
The first pair gives a resultant $\vec{F}_1 = \vec{A} + \vec{B}$ that has components $F_{1,x} = -1$ N and $F_{1,y} = +7$ N, and magnitude $F_1 = 7.07$ N.

The second pair gives a resultant $\vec{F}_2 = \vec{B} + \vec{C}$ that has components $F_{2,x} = -3$ N and $F_{2,y} = -2$ N, and magnitude $F_2 = 3.61$ N.

The last pair gives a resultant $\vec{F}_3 = \vec{C} + \vec{A}$ that has components $F_{3,x} = +4$ N and $F_{3,y} = +1$ N, and magnitude $F_3 = 4.12$ N.

The pair with the resultant of greatest magnitude is $\vec{A}$ and $\vec{B}$.

Exr 0.2.21



With three vectors, there are three possible pairs.

The first pair gives a resultant $\vec{F}_1 = \vec{A} + \vec{B}$ that has components $F_{1,x} = 0\,\text{N}$ and $F_{1,y} = +8\,\text{N}$, and magnitude $F_1 = 8\,\text{N}$.

The second pair gives a resultant $\vec{F}_2 = \vec{B} + \vec{C}$ that has components $F_{2,x} = -5\,\text{N}$ and $F_{2,y} = -7\,\text{N}$, and magnitude $F_2 = 8.60\,\text{N}$.

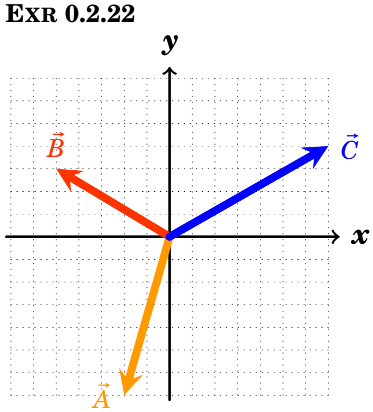
The last pair gives a resultant $\vec{F}_3 = \vec{C} + \vec{A}$ that has components $F_{3,x} = +3\,\text{N}$ and $F_{3,y} = +7\,\text{N}$, and magnitude $F_3 = 7.62\,\text{N}$.

The pair with the resultant of greatest magnitude is $\vec{B}$ and $\vec{C}$. (Note how the resultant of the two largest vectors does not have the largest magnitude.)

0.2.3 Vectors that Sum to Zero

In the exercises below each vector is a force, and each grid square corresponds to one newton (1 N) of force. For the sets of vectors below find the missing vector that would make their sum equal $\vec{0}\,\text{N}$. (Remember that the vector $\vec{0}\,\text{N}$ is the vector whose components are each 0 N.) In the case of two vectors being given (which we can call $\vec{A}$ and $\vec{B}$) find the third vector $\vec{C}$ such that $\vec{A} + \vec{B} + \vec{C} = \vec{0}\,\text{N}$.

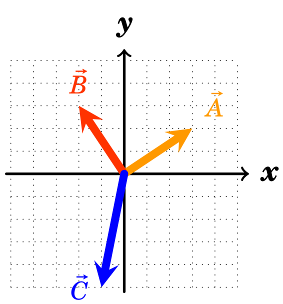
Exr 0.2.22



Given that $A_x = -2\,\text{N}$, $A_y = -7\,\text{N}$, $B_x = -5\,\text{N}$, and $B_y = +3\,\text{N}$, we can solve the vector equation $\vec{A} + \vec{B} + \vec{C} = \vec{0}\,\text{N} \longrightarrow \vec{C} = -(\vec{A} + \vec{B})$ by its components

$$
\begin{aligned}
C_x &= -(A_x + B_x) && (49)\\
&= -\big((-2\,\text{N}) + (-5\,\text{N})\big) = +7\,\text{N} && (50)\\
C_y &= -(A_y + B_y) && (51)\\
&= -\big((-7\,\text{N}) + (+3\,\text{N})\big) = +4\,\text{N} && (52)
\end{aligned}
$$

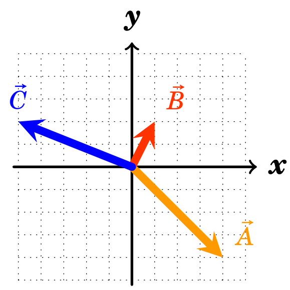
Exr 0.2.23



Given that $A_x = +3\,\text{N}$, $A_y = +2\,\text{N}$, $B_x = -2\,\text{N}$, and $B_y = +3\,\text{N}$, we can solve the vector equation $\vec{A} + \vec{B} + \vec{C} = \vec{0}\,\text{N} \longrightarrow \vec{C} = -(\vec{A} + \vec{B})$ by its components

$$
\begin{aligned}
C_x &= -(A_x + B_x) && (53)\\
&= -\big((+3\,\text{N}) + (-2\,\text{N})\big) = -1\,\text{N} && (54)\\
C_y &= -(A_y + B_y) && (55)\\
&= -\big((+2\,\text{N}) + (+3\,\text{N})\big) = -5\,\text{N} && (56)
\end{aligned}
$$

Exr 0.2.24



Given that $A_x = +4\,\text{N}$, $A_y = -4\,\text{N}$,
$B_x = +1\,\text{N}$, and $B_y = +2\,\text{N}$,
we can solve the vector equation
$\vec{A} + \vec{B} + \vec{C} = \vec{0}\,\text{N} \longrightarrow \vec{C} = -(\vec{A} + \vec{B})$
by its components

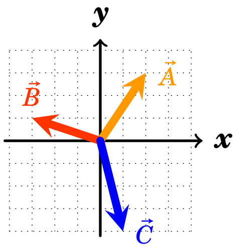
$$C_x = -(A_x + B_x) \tag{57}$$

$$= -\big((+4\,\text{N}) + (+1\,\text{N})\big) = -5\,\text{N} \tag{58}$$

$$C_y = -(A_y + B_y) \tag{59}$$

$$= -\big((-4\,\text{N}) + (+2\,\text{N})\big) = +2\,\text{N} \tag{60}$$

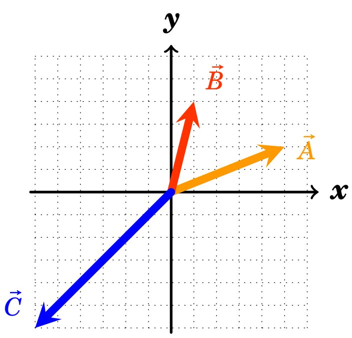
Exr 0.2.25



Given that $A_x = +2\,\text{N}$, $A_y = +3\,\text{N}$,
$B_x = -3\,\text{N}$, and $B_y = +1\,\text{N}$,
we can solve the vector equation
$\vec{A}+\vec{B}+\vec{C}=\vec{0}\,\text{N} \longrightarrow \vec{C}=-(\vec{A}+\vec{B})$
by its components

$$\begin{aligned} C_x &= -(A_x+B_x) && (61)\\ &= -\big((+2\,\text{N})+(-3\,\text{N})\big) = +1\,\text{N} && (62)\\ C_y &= -(A_y+B_y) && (63)\\ &= -\big((+3\,\text{N})+(+1\,\text{N})\big) = -4\,\text{N} && (64)\end{aligned}$$

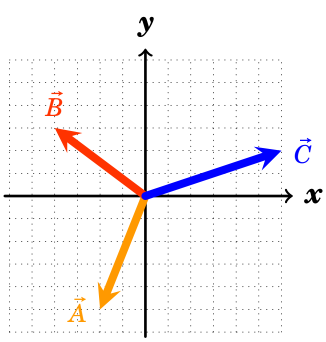
Exr 0.2.26



Given that $A_x = +5\,\text{N}$, $A_y = +2\,\text{N}$,
$B_x = +1\,\text{N}$, and $B_y = +4\,\text{N}$,
we can solve the vector equation
$\vec{A}+\vec{B}+\vec{C}=\vec{0}\,\text{N} \longrightarrow \vec{C}=-(\vec{A}+\vec{B})$
by its components

$$\begin{aligned} C_x &= -(A_x+B_x) && (65)\\ &= -\big((+5\,\text{N})+(+1\,\text{N})\big) = -6\,\text{N} && (66)\\ C_y &= -(A_y+B_y) && (67)\\ &= -\big((+2\,\text{N})+(+4\,\text{N})\big) = -6\,\text{N} && (68)\end{aligned}$$

Exr 0.2.27



Given that $A_x = -2\,\text{N}$, $A_y = -5\,\text{N}$,
$B_x = -4\,\text{N}$, and $B_y = +3\,\text{N}$,
we can solve the vector equation
$\vec{A}+\vec{B}+\vec{C}=\vec{0}\,\text{N} \longrightarrow \vec{C}=-(\vec{A}+\vec{B})$
by its components

$$\begin{aligned} C_x &= -(A_x+B_x) && (69)\\ &= -\big((-2\,\text{N})+(-4\,\text{N})\big) = +6\,\text{N} && (70)\\ C_y &= -(A_y+B_y) && (71)\\ &= -\big((-5\,\text{N})+(+3\,\text{N})\big) = +2\,\text{N} && (72)\end{aligned}$$

0.3 Logarithms

The square-root function tells you "if $x=\sqrt{y}$, then $y=x^2$." In a similar way the *logarithm* function tells you "if $x=\log(y)$, then $y=10^x$." The logarithm (or just "log" for short) has some *algebraic* properties. You know that for the square-root $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$. For the log there are the rules $\log(a\times b)=\log(a)+\log(b)$ and $\log(c^n)=n\times\log(c)$. The exercises in this section practice these properties.

0.3.1 Basics

Exr 0.3.01 $\log(100) = 2$
Exr 0.3.02 $\log(1\,000) = 3$
Exr 0.3.03 $\log(500) = 2.70$
Exr 0.3.04 $\log(50) = 1.70$
Exr 0.3.05 $\log(5) = 0.7$
Exr 0.3.06 $\log(1\,000\,000) = 6$
Exr 0.3.07 $\log(10\,000\,000) = 7$

Exr 0.3.08 $\log(3\,000\,000) = 6.477$
Exr 0.3.09 $\log(9.900\times10^5) = 5.996$
Exr 0.3.10 $\log(100\times1000) = 5$
Exr 0.3.11 $\log(100)+\log(1000) = 5$
Exr 0.3.12 $\log(15) = 1.176$
Exr 0.3.13 $\log(3)+\log(5) = 1.176$
Exr 0.3.14 $\log(1000/10) = 2$

Exr 0.3.15 $\log(1000) - \log(10) = 2$
Exr 0.3.16 $\log(15) = 1.176$
Exr 0.3.17 $\log(30) - \log(2) = 1.176$
Exr 0.3.18 $\log(7/3) = 0.368$
Exr 0.3.19 $\log(7) - \log(3) = 0.368$
Exr 0.3.20 $\log(3/7) = -0.368$
Exr 0.3.21 $\log(137/137) = 0$
Exr 0.3.22 $\log(\frac{1}{100}) = -2$
Exr 0.3.23 $\log(5) = 0.7$
Exr 0.3.24 $\log(25) = 1.4$
Exr 0.3.25 $2 \times \log(5) = 1.4$
Exr 0.3.26 $\log(2) = 0.3$
Exr 0.3.27 $\log(8) = 0.9$
Exr 0.3.28 $3 \times \log(2) = 0.9$
Exr 0.3.29 $\log(\sqrt{10}) = 1/2$
Exr 0.3.30 $\log(\sqrt{\sqrt{10}}) = 1/4$

0.3.2 Solving Equations with Logarithms

One last note: Since $10^x > 0$ for any value of x, there is no "$\pm$" when we use the logarithm to solve an equation. Also for that reason expressions like $\log(-7)$ have no meaning (just like $\sqrt{-7}$ is not a number).

Exr 0.3.31 If $10^x = 10^3$, then $x = +3$.
Exr 0.3.32 If $10^x = 10^{-5}$, then $x = -5$.
Exr 0.3.33 If $10^x = 10^7$, then $x = +7$.
Exr 0.3.34 If $10^x = 10^{-2}$, then $x = -2$.
Exr 0.3.35 If $10^x = 10^{3/4}$, then $x = +3/4$.
Exr 0.3.36 If $10^x = 10^{-11/7}$, then $x = -11/7$.
Exr 0.3.37 If $10^x = 10^{-37/81}$, then $x = -37/81$.
Exr 0.3.38 If $10^x = 10^{5/2}$, then $x = +5/2$.
Exr 0.3.39 If $10^x = 10^{2.957}$, then $x = +2.957$.
Exr 0.3.40 If $10^x = 10^{\sqrt{2}}$, then $x = +\sqrt{2}$.
Exr 0.3.41 If $10^x = 10^{-0.8251}$, then $x = -0.8251$.
Exr 0.3.42 If $10^x = 10^{\pi}$, then $x = \pi$.

Exr 0.3.43 If $\log(x) = +4$, then $x = 10^{+4} = 10\,000$.
Exr 0.3.44 If $\log(x) = +1$, then $x = 10^{+1} = 10$.
Exr 0.3.45 If $\log(x) = -5$, then $x = 10^{-5} = 0.000\,01$.
Exr 0.3.46 If $\log(x) = -2$, then $x = 10^{-2} = 0.01$.
Exr 0.3.47 If $\log(x) = -3/2$, then $x = 10^{-3/2} = 0.031\,623$.
Exr 0.3.48 If $\log(x) = 4/11$, then $x = 10^{+4/11} = 2.310\,130$.
Exr 0.3.49 If $\log(x) = 1\,\frac{1}{7}$, then $x = 10^{+8/7} = 13.894\,955$.
Exr 0.3.50 If $\log(x) = 2.718\,282$, then $x = 10^{2.718\,282} = 522.736$.
Exr 0.3.51 If $\log(x) = \sqrt{3}$, then $x = 10^{\sqrt{3}} = 53.957\,374$.
Exr 0.3.52 If $\log(x) = -\pi$, then $x = 10^{-\pi} = 0.000\,721\,784$.

Chapter 1

Forces

1.1 Free-Body Diagrams: Forces in Static Equilibrium

In these exercises we will practice the construction and use of Free-Body Diagrams (FBDs) to reason about the forces acting on objects in static equilibrium. For the purposes of practice we will limit ourselves to two-dimensional systems. Unless explicitly stated each system is being viewed from the side so that gravity acts straight downwards on the diagram.

In each exercise construct the Free-Body Diagram (FBD). A recommended first step is to make a list of the things that the object is interacting with. Name those things, beginning with the Earth, and then name the other objects or surfaces which the object is touching (like the floor) or is attached to (like a rope). Look carefully at the diagram (if given), and read carefully any descriptive text of the situation (if given), during this step.

After constructing the FBD, check that the forces do sum to zero. This check is a qualitative diagram used to confirm that equilibrium is *possible*. If it is not possible to sum the forces in your FBD by adjusting them, then return to your list of interactions, and think more carefully about the situation. (There is no point to move past this step to doing quantitative calculations if the required outcome is impossible!)

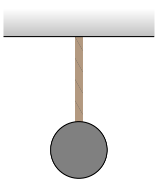
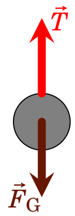
Remember to be clear about what aspect of each force is unknown. If there is a rope at a specified angle, then the force of tension that it exerts on the object is along that fixed direction, and only its magnitude may be adjusted. If there is a surface of contact, then the *normal* at that surface points perpendicular away from the surface, and only its magnitude may be adjusted. Similarly, if there is friction, then it must be parallel to the surface, but its direction and magnitude must both be determined.

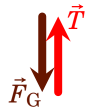
Above all, do not hesitate to *iterate*. Producing the “correct” diagram immediately is not a healthy expectation to hold. You are solving a problem and you must be open to exploring alternatives before arriving at a consistent answer. Iteration is a key feature of problem-solving!

1.1.1 Gravity and Tension

In these first few Free-Body Diagram (FBD) exercises the object is suspended against gravity by ropes. In each exercise, draw the FBD, and use the sum of forces to *qualitatively* estimate the magnitudes of the tensions (relative to your choice of the weight of the object).

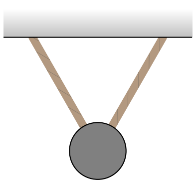
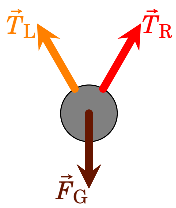
EXERCISE 1.1.01

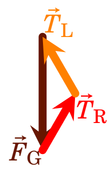





Physically: The tension must balance the weight, and so has an equal magnitude, as is expected.

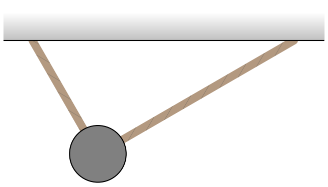
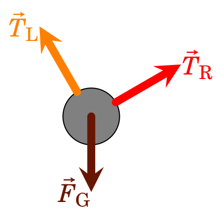
EXERCISE 1.1.02

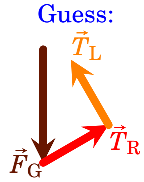


Physically: The magnitude of the tension in each rope is slightly greater than half the weight because they are pulling horizontally against each other in addition to supporting half the weight.

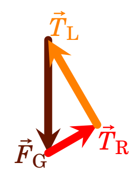
EXERCISE 1.1.03

Guess:

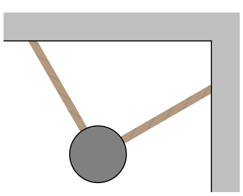
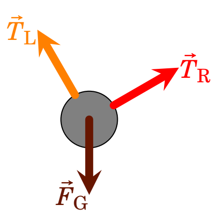


The tension on the right should be weaker, and the tension on the left slightly stronger, relative to the weight.

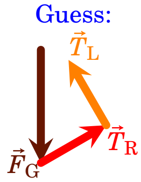


Physically: The rope on the left, that is more vertical, supports more of the weight and thus has a greater magnitude of tension than the rope on the right.

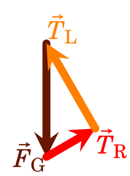
EXERCISE 1.1.04

Guess:

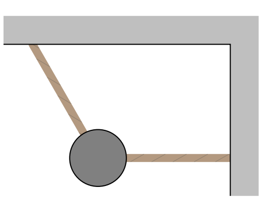
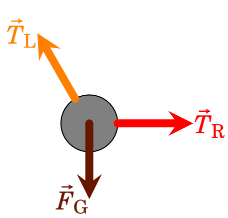


The lengths of the ropes are different, but the forces are the same as the previous exercise.

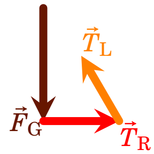


Physically: The length of a rope and the tension in the rope are independent of each other. The angles made by the ropes are the same as the previous exercise, and so the tensions are the same as in the previous exercise.

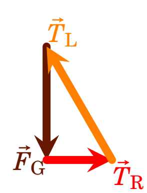
EXERCISE 1.1.05

Guess:

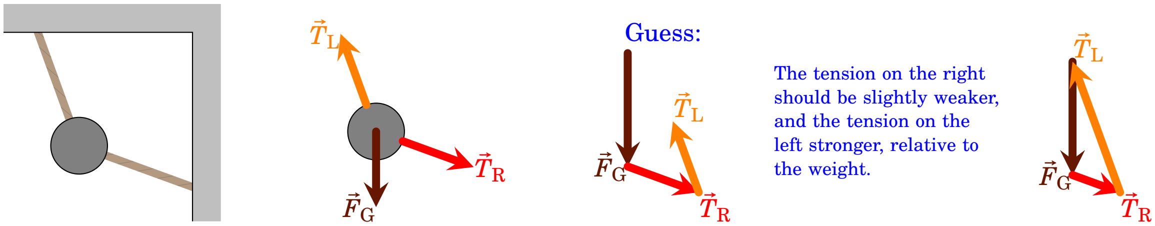


The tension on the left should be stronger, relative to the weight.



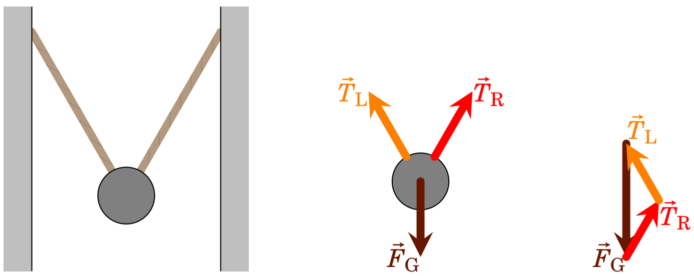
Physically: The tension in the horizontal rope is independent of the object's weight. The rope on the left must support the weight and oppose the rope on the right, and so is greater in magnitude than either.

EXERCISE 1.1.06



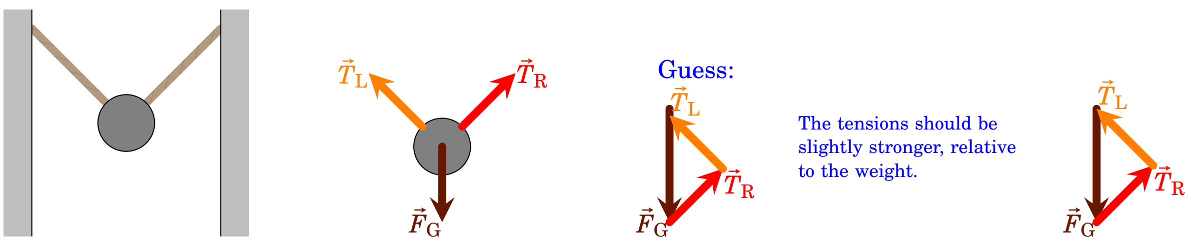
Physically: The rope above and towards the left must support the weight and counter the tension in the rope below and towards the right. The weight is only pulled away from the vertical by a small angle, so the tension on the right does not need to be large.

EXERCISE 1.1.07



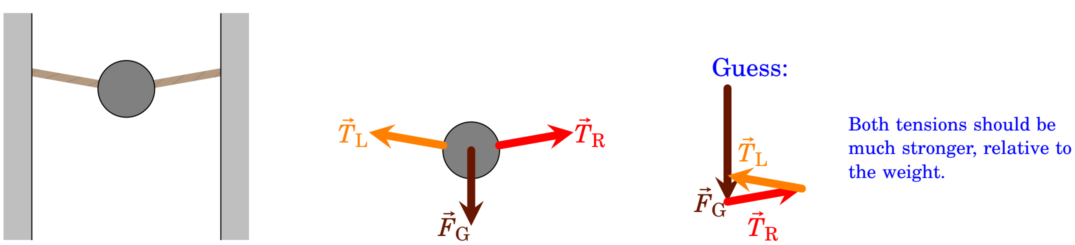
Physically: The magnitude of the tension in each rope is slightly greater than half the weight because they are pulling horizontally against each other in addition to supporting half the weight.

EXERCISE 1.1.08

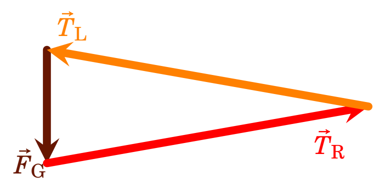


Physically: The magnitude of the tension in each rope is greater than half the weight because they are pulling horizontally against each other in addition to supporting half the weight.

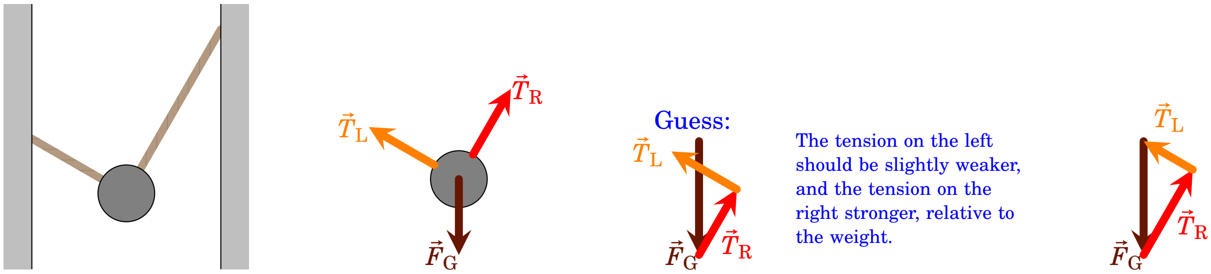
EXERCISE 1.1.09



Physically: The vertical components of the tensions in the ropes must balance the weight. Since they are both so close to being horizontal they must both be large in magnitude to achieve the required vertical balance.

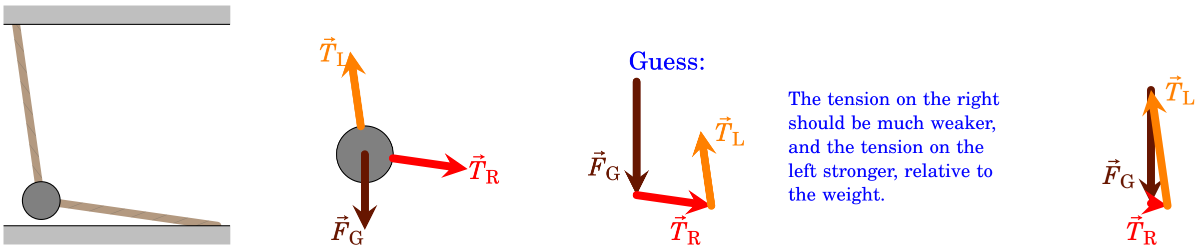


EXERCISE 1.1.10



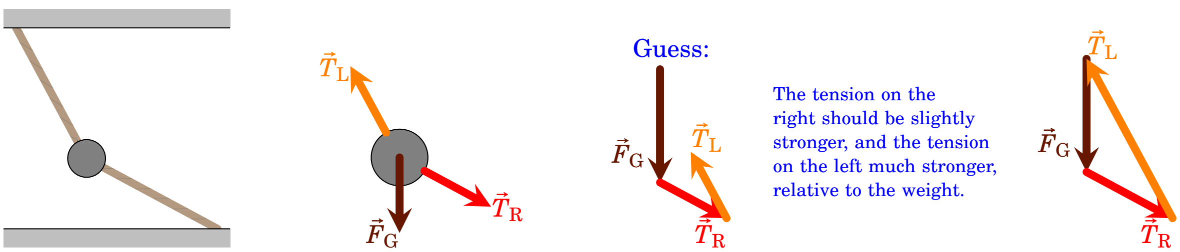
Physically: The rope on the right, that is more vertical, supports more of the weight and thus has a greater magnitude of tension than the rope on the left.

Exercise 1.1.11



Physically: The rope above must support the weight and counter the tension in the rope towards the right. The weight is only pulled away from the vertical by very a small angle, so the tension on the right does not need to be large.

Exercise 1.1.12

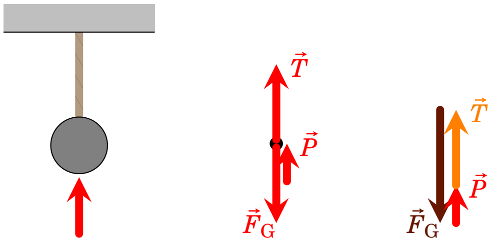


Physically: The rope above must support the weight and counter the tension in the rope towards the right. The ropes are almost straight, aligned with each other, so the tensions must be larger than the weight.

1.1.2 With an External Force

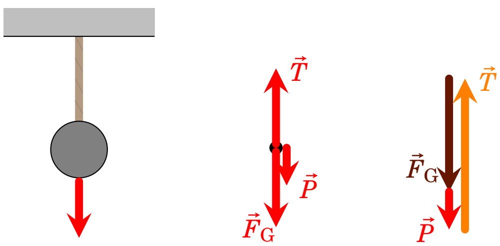
In these cases an external force $\vec{P}$ (a push or a pull) is being applied to an object suspended by ropes. In each of these cases the applied force never causes the tension of any rope to become zero.

Exercise 1.1.13



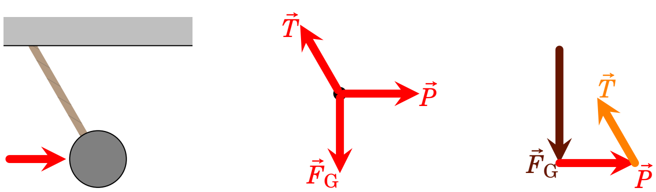
Physically: The tension must balance the weight and the applied push. Since the push is upwards, this reduces the required tension.

Exercise 1.1.14

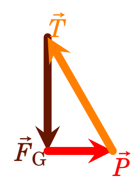


Physically: The tension must balance the weight and the applied pull. Since the pull is downwards, this increases the required tension.

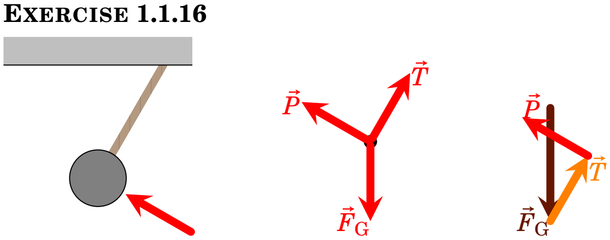
Exercise 1.1.15



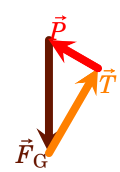
The tension int he rope should be slightly stronger, relative to the weight.



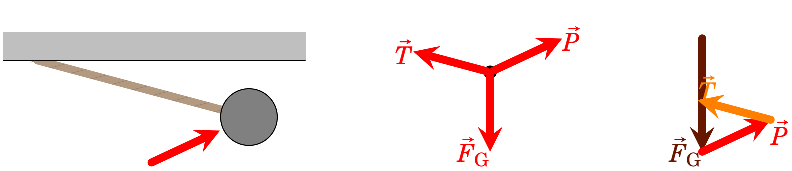
Exercise 1.1.16



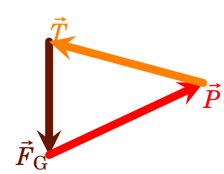
We have over-estimated the strength of the push relative to the weight. The tension should be stronger, relative to the weight.



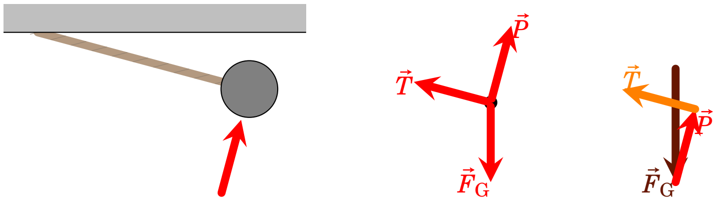
Exercise 1.1.17



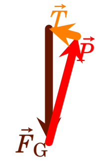
The applied push and the tension should be stronger, relative to the weight.



Exercise 1.1.18

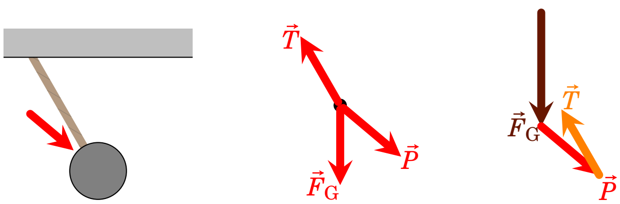


We have over-estimated the tension relative to the weight. The push should be stronger, relative to the weight.

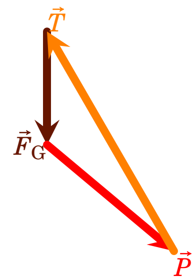


Physically: The applied push is almost vertical, and so supports most of the weight. The tension does not need to be very large to keep the object in static equilibrium.

EXERCISE 1.1.19

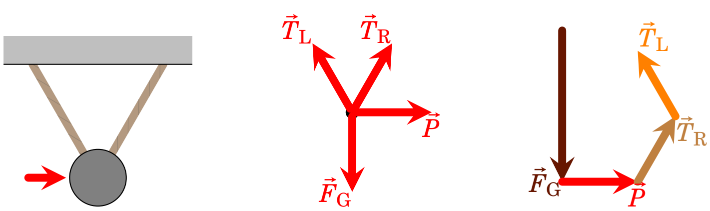


Both the tension and the push should be much stronger, relative to the weight.

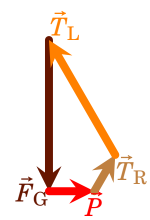


Physically: Since the applied force is not horizontal it must be large to hold the object away from the vertical. The applied force, almost parallel to the rope, increases the tension significantly.

EXERCISE 1.1.20

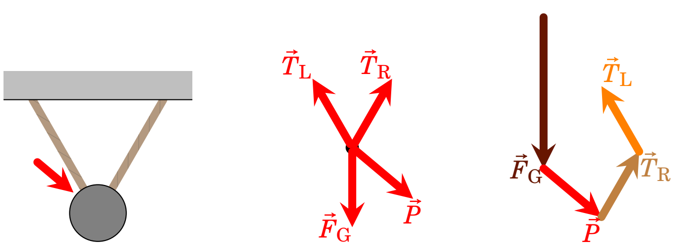


Both the tension on the right and the push should be weaker, and the tension on the left should be slightly stronger, relative to the weight.

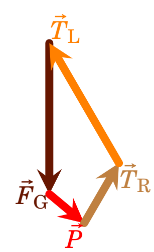


Physically: Since the applied force is horizontal the rope that can oppose it (the rope on the left) must have a higher tension. This also decreases the tension in the rope on the right.

EXERCISE 1.1.21

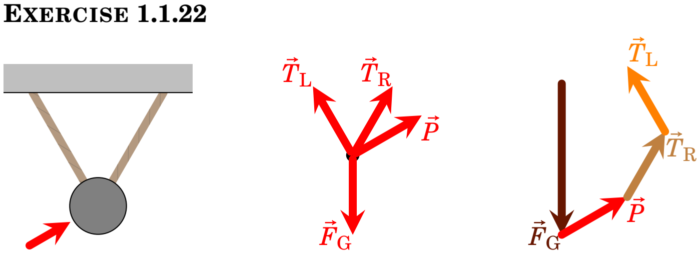


Both the tension on the right and the push should be weaker, and the tension on the left should be slightly stronger, relative to the weight.

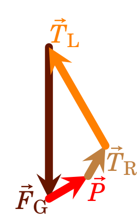


Physically: Since the applied force is almost parallel to the rope on the left that rope must have an increased tension. This also decreases the tension in the rope on the right.

EXERCISE 1.1.22

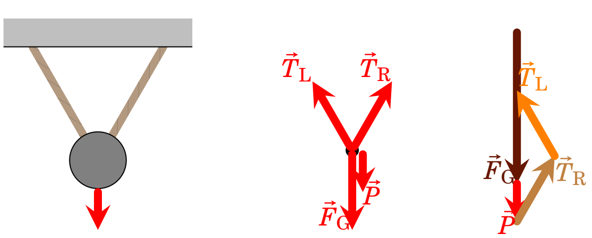


Both the tension on the right and the push should be much weaker, and the tension on the left should be stronger, relative to the weight.

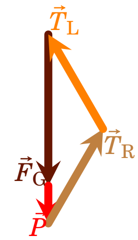


Physically: Since the applied force is perpendicular to the rope on the left the tension in that rope will not be changed relative to the situation without the push. Since the push is upwards and towards the right it supports some of the weight, and reduces the tension on the right.

Exercise 1.1.23

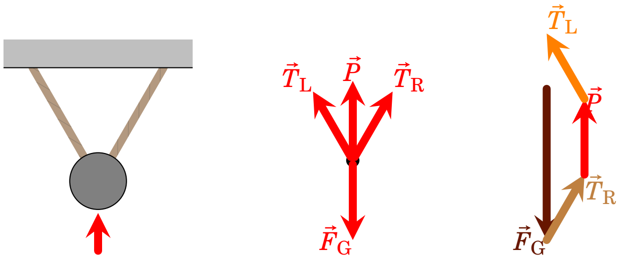


Both tensions should be stronger, relative to the weight.

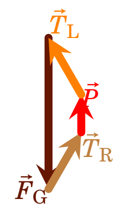


Physically: The applied force contributes downwards, with gravity. This increases the tension in both ropes, symmetrically.

Exercise 1.1.24

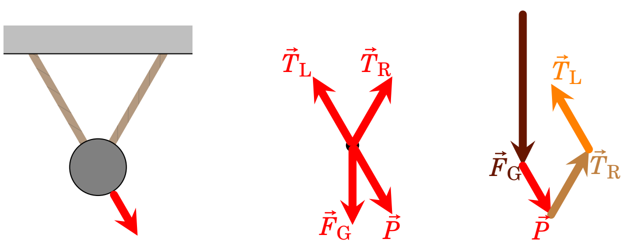


Both tensions should be weaker, relative to the weight.

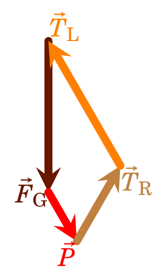


Physically: Since the applied force supports a part of the weight the tension in both ropes is decreased.

Exercise 1.1.25

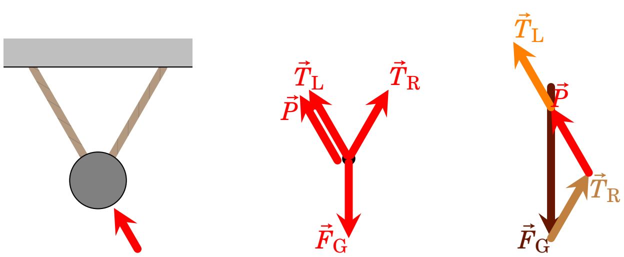


Both the tension on the left should be stronger, relative to the weight.

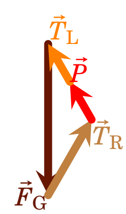


Physically: Since the applied force is parallel to the rope on the left that rope must have an increased tension. This does not change the tension in the rope on the right.

Exercise 1.1.26



Both the tension on the left and the push should be weaker, relative to the weight.



Physically: Since the applied force is parallel to the rope on the left that rope must have a decreased tension. This does not change the tension in the rope on the right.

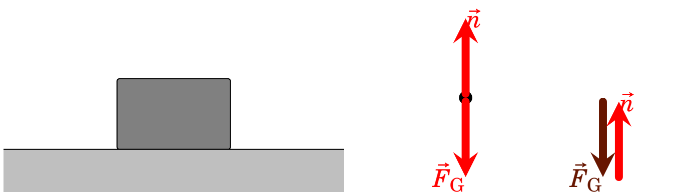
1.1.3 Gravity and Contact

In this section we will practice analyzing objects in contact with a surface. Friction is present between all surfaces of contact. The *exceptions* of there being negligible friction will be noted as $\mu = 0$.

Level Surfaces

In these situations the force of gravity and the *normal* will point opposite each other, but, because of other forces present, they may not be of equal magnitude. Careful!

EXERCISE 1.1.27

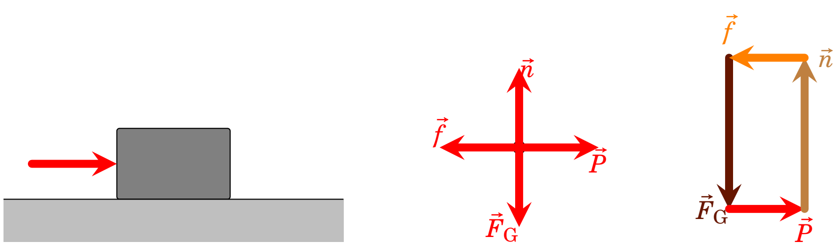


Physically: The normal must balance the weight, and so has an equal magnitude, as is expected.

EXERCISE 1.1.28

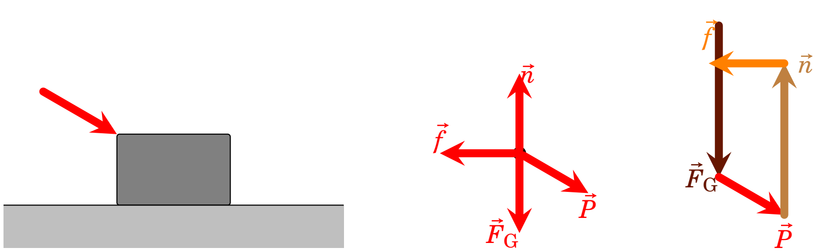
Physically: The normal must balance the weight and the applied push. Since the push is downwards, this increases the required normal.

EXERCISE 1.1.29

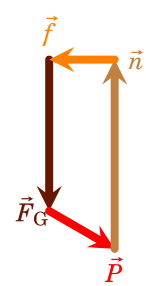


Physically: Friction opposes the push.

EXERCISE 1.1.30

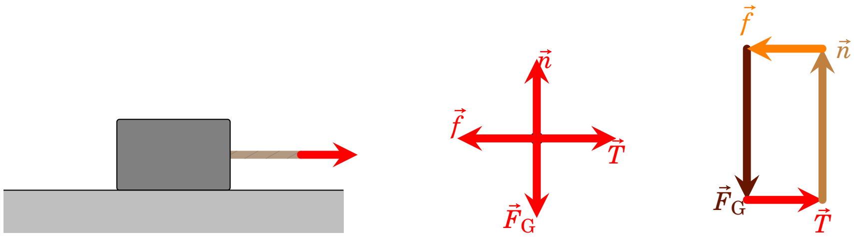


The normal should be larger, relative to the weight. The friction should be slightly smaller, relative to the push.



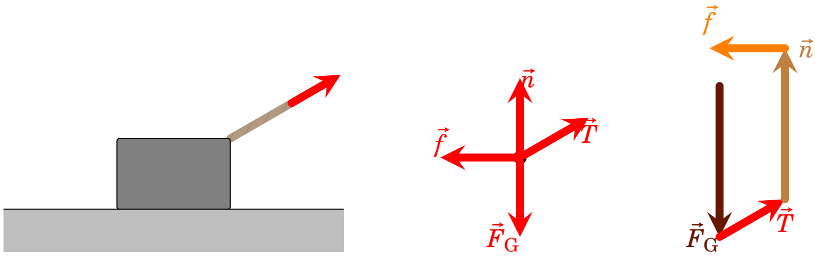
Physically: The normal is larger than the weight since it must also oppose the vertical component of the applied push. Friction opposes the horizontal component of the applied push.

EXERCISE 1.1.31

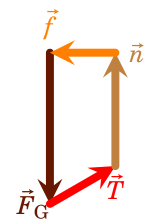


Physically: Friction opposes the pull applied by the tension in the rope.

EXERCISE 1.1.32

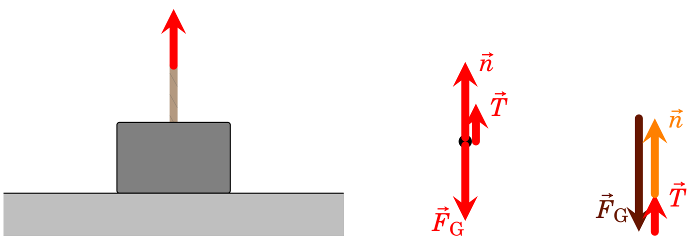


The normal should be smaller, relative to the weight. The friction should be slightly smaller, relative to the tension.



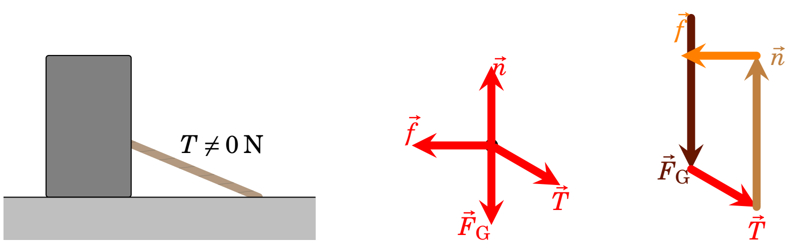
Physically: The normal is smaller than the weight since the vertical component of the applied tension supports a portion of the weight. Friction opposes the horizontal component of the tension.

EXERCISE 1.1.33

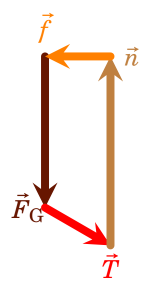


Physically: The normal must balance the weight and the applied tension. Since the tension is upwards, this decreases the required normal.

EXERCISE 1.1.34 An object is attached to the surface by a rope. There is tension in the rope.



The normal should be larger, relative to the weight. The friction should be slightly smaller, relative to the tension.

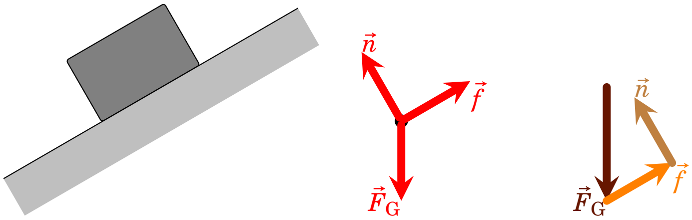


Physically: The normal is larger than the weight since it must also oppose the vertical component of the tension that is trying to pull the object towards the surface. Friction opposes the horizontal component of the tension that is trying to pull the object the towards the right.

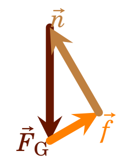
Inclined Surfaces

In these exercises friction remains strong enough to keep the object in equilibrium on the surface. The *normal* is still (as always) perpendicular to the surface, but with the surface not horizontal the normal will *not* point opposite gravity, and will almost certainly not have the same magnitude. Be very careful finding the normal!

EXERCISE 1.1.35

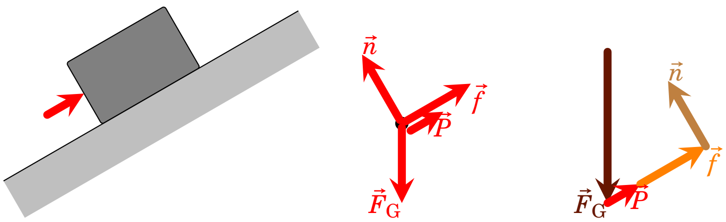


We have over-estimated the strength of the push relative to the weight. The tension should be stronger, relative to the weight.

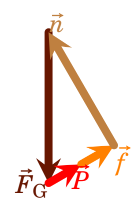


Physically: The normal prevents the object from moving through the surface, but it is of the wrong magnitude and direction to counter gravity. The force of friction completes the sum to zero, preventing the object from moving across the surface.

EXERCISE 1.1.36 The applied force is small in comparison to all other forces.

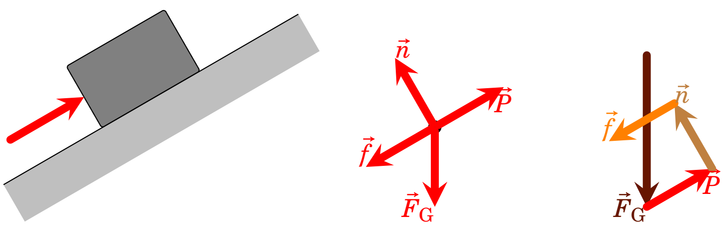


We have over-estimated the strength of friction, and the normal should be stronger, relative to the weight.

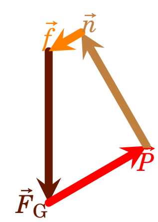


Physically: The push is insufficient to counter the component of gravity parallel to the surface. Thus friction must still point upwards along the surface.

EXERCISE 1.1.37 The applied force is large, and the object almost starts sliding UP the incline.

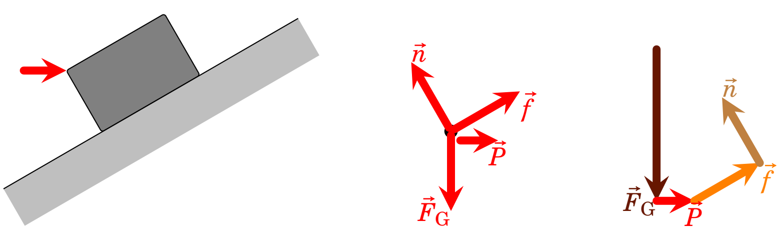


We have under-estimated the strength of the push and the normal, relative to the weight.

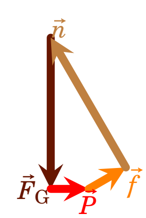


Physically: We are told that the object is almost about to begin sliding *up* the incline. The push upwards along the incline is more than required to support the component of the weight parallel to the surface, so friction must oppose it. This means that friction must point *down* the incline to oppose that incipient motion.

EXERCISE 1.1.38 The applied force is small in comparison to all other forces.

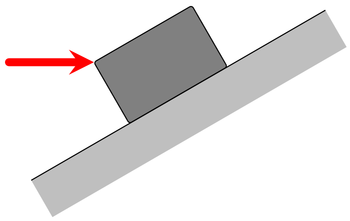


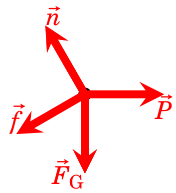
We have over-estimated the strength of friction, and the normal should be stronger, relative to the weight.

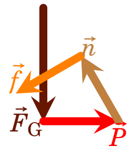


Physically: The push, being into the surface and upwards along the incline, decreases friction and increases the normal.

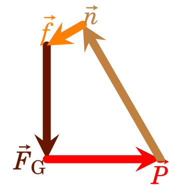
EXERCISE 1.1.39 The applied force is large, and the object almost starts sliding UP the incline.





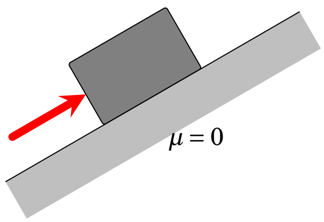


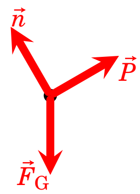
We have under-estimated the strength of the push and the normal, and over-estimated the strength of friction, relative to the weight.

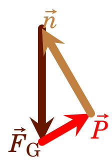


Physically: The push, being strongly into the surface and upwards along the incline, reverses the direction of friction, and increases the normal significantly.

Exercise 1.1.40

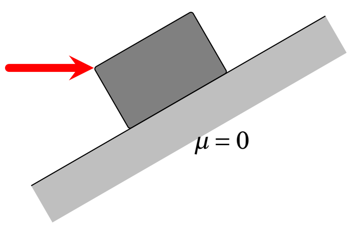


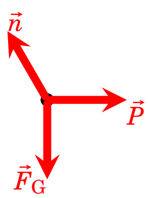


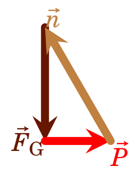


Physically: The normal prevents the object from moving through the surface, but it is of the wrong magnitude and direction to counter gravity. The applied push, in the place of friction, completes the sum to zero, preventing the object from moving across the surface.

Exercise 1.1.41

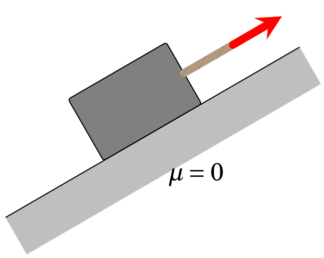


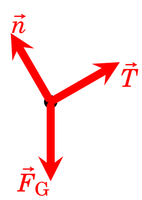


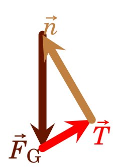


Physically: The applied push, directed into and upwards along the incline, increases the normal and balances the component of gravity parallel to the surface.

Exercise 1.1.42

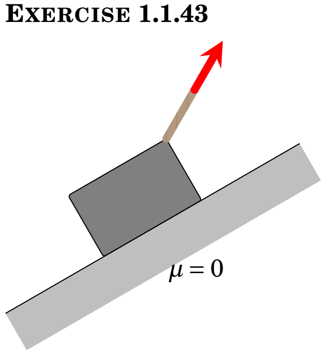


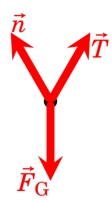


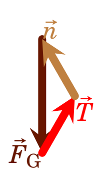


Physically: The applied tension replaces friction as the balancing force.

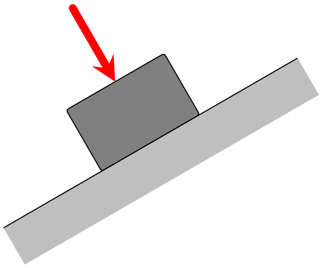
Exercise 1.1.43

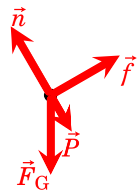


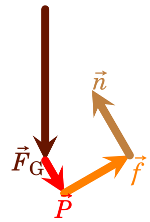




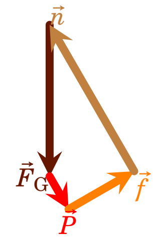
EXERCISE 1.1.44





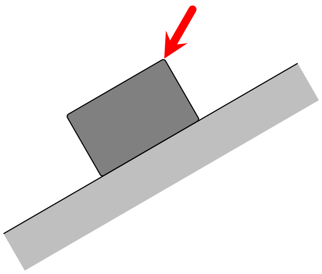


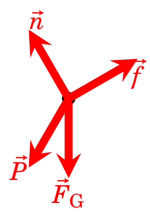
The normal should be stronger, relative to the weight.

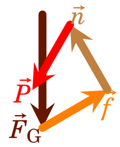


Physically: The push, directed perpendicular to the surface, only increases the strength of the normal required to stop the object from moving through the surface.

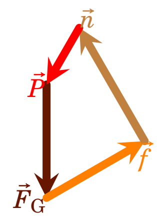
EXERCISE 1.1.45





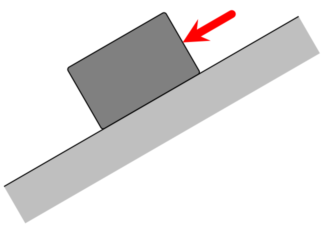


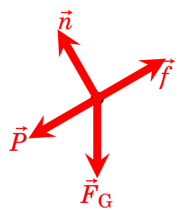
We have under-estimated the strength of friction and the normal, relative to the weight.

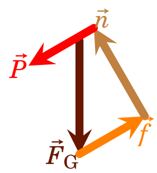


Physically: The push, being into the surface and downwards along the incline, increases both friction and the normal.

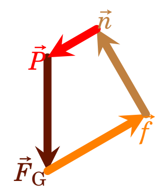
EXERCISE 1.1.46







We have under-estimated the strength of friction, relative to the push.

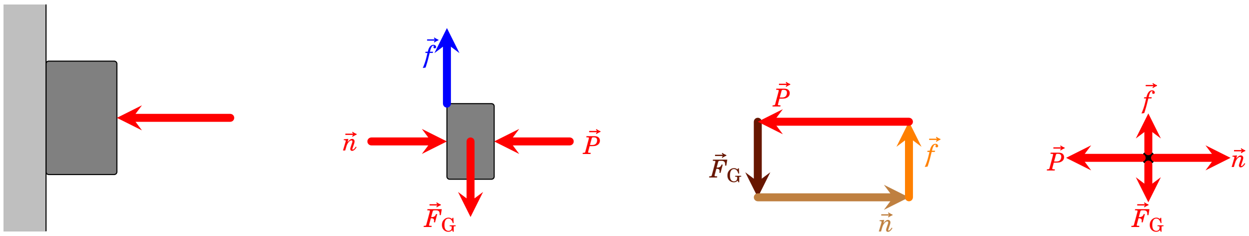


Physically: The push, being downwards along the incline, increases the friction.

Cases of vertical and inverted surface

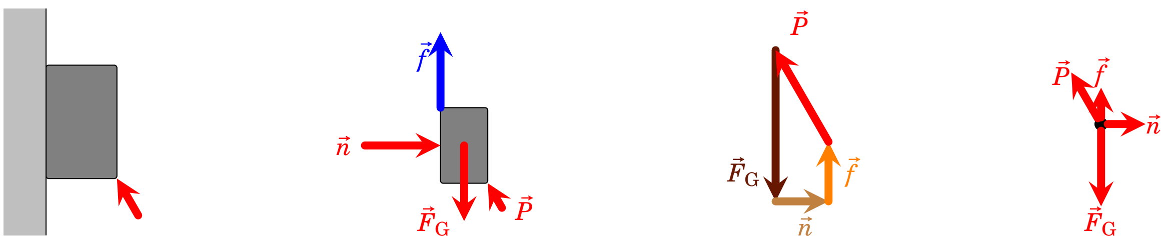
The theme to recognize in these exercises is that, with the surface vertical or even *upside-down*, the normal force can not contribute to supporting the object's weight, and may in fact be contributing a downwards component! Look to the applied force(s) and friction (when present) to support the object against gravity.

EXERCISE 1.1.47 The applied force has a magnitude *much larger* than the block's weight.



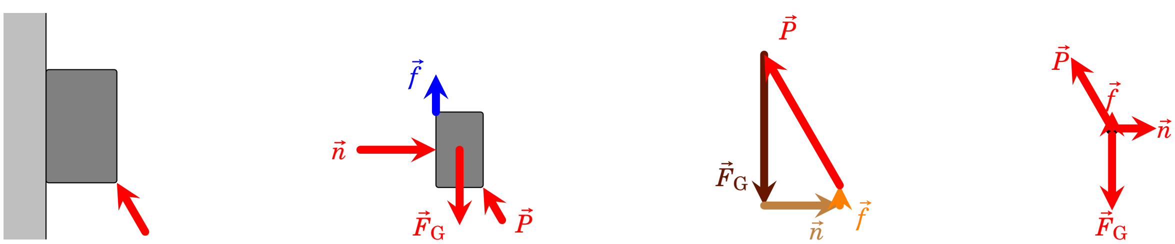
Physically: The normal balances the applied force. From the check we see that friction must point *upwards* to prevent the block from sliding down the surface.

EXERCISE 1.1.48 The applied force has a magnitude *smaller* than the block's weight.



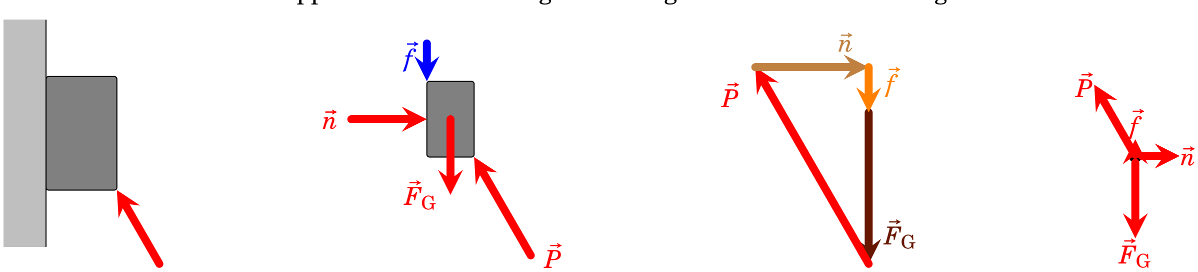
Physically: The normal balances the horizontal portion of the applied force. From the check we see that friction must make an *upwards* contribution to prevent the block from sliding down the surface.

EXERCISE 1.1.49 The applied force has a magnitude *equal* to the block's weight.



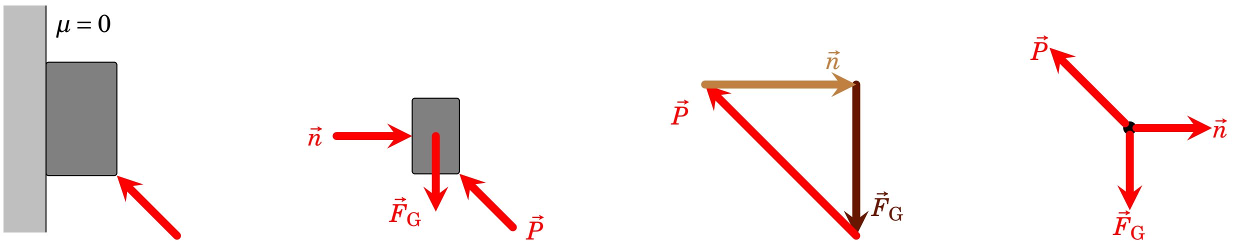
Physically: The normal balances the horizontal portion of the applied force. From the check we see that friction must make a small *upwards* contribution to prevent the block from sliding down the surface.

EXERCISE 1.1.50 The applied force has a magnitude *larger* than the block's weight.



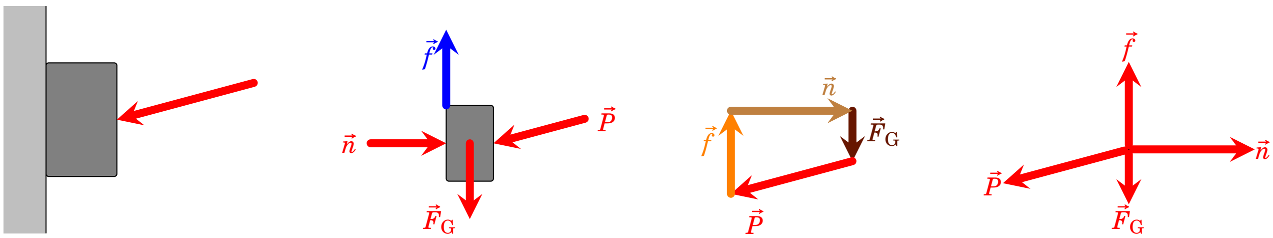
Physically: The normal balances the horizontal portion of the applied force. From the check we see that, since the push is larger than the weight, friction must make a small *downwards* contribution to prevent the block from sliding *up* the surface.

EXERCISE 1.1.51 No friction between the block and surface.



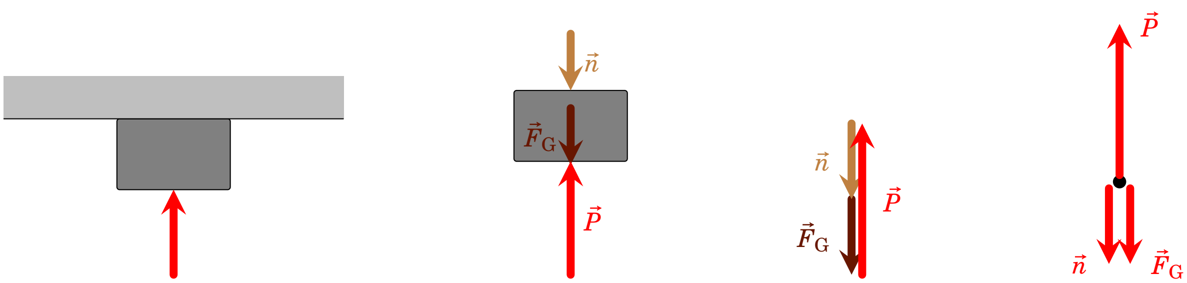
Physically: With no friction present the three forces in the sum form a simple triangle. Without friction to help support the object, the vertical component of the push must balance the weight.

EXERCISE 1.1.52 The applied force has a magnitude *much larger* than the block's weight.



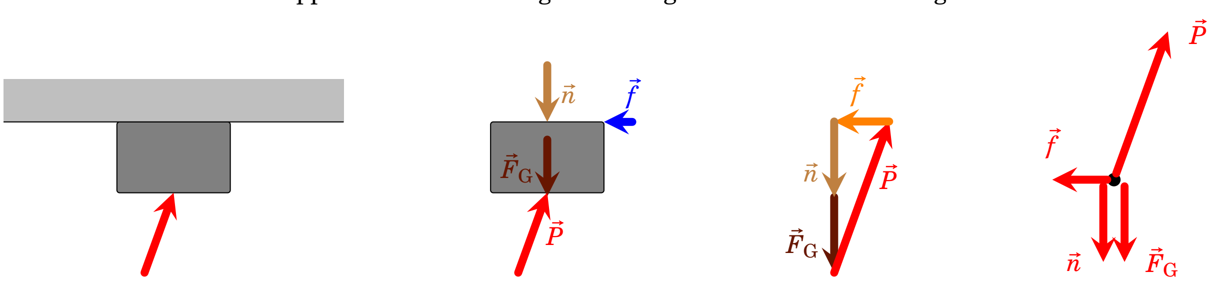
Physically: The normal balances the vertical portion of the applied force. From the check we see that friction must point *upwards* to support the block against the combination of gravity and the applied force (which is directed slightly downwards).

EXERCISE 1.1.53 The applied force has a magnitude *larger* than the block's weight.



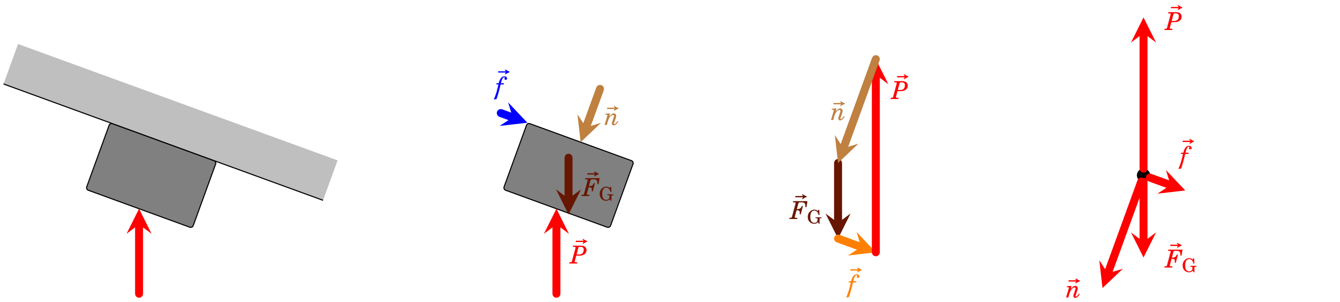
Physically: The applied force, which is larger than the object's weight, is pressing the object into the surface. The normal points *downwards* (away from the surface) to oppose this.

EXERCISE 1.1.54 The applied force has a magnitude *larger* than the block's weight.



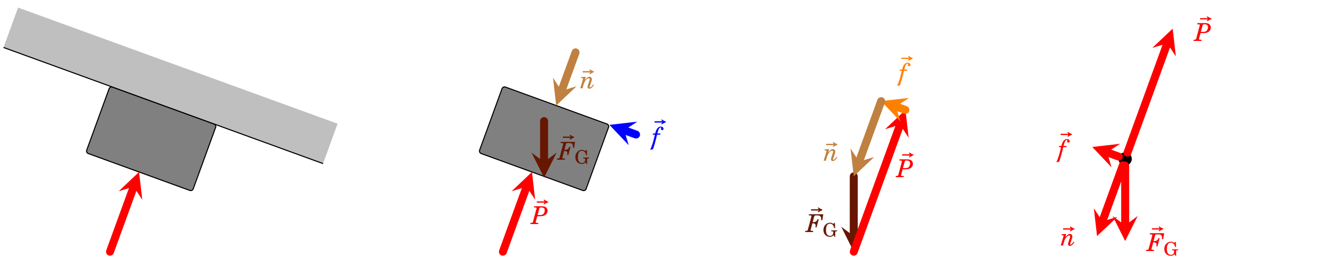
Physically: The applied force, which is larger than the object's weight, is pressing the object into the surface. The normal points *downwards* (away from the surface) to oppose this. There is a portion of the applied force that is parallel to the surface, and friction opposes this.

EXERCISE 1.1.55 The applied force is vertically upwards, and has a magnitude *larger* than the block's weight.



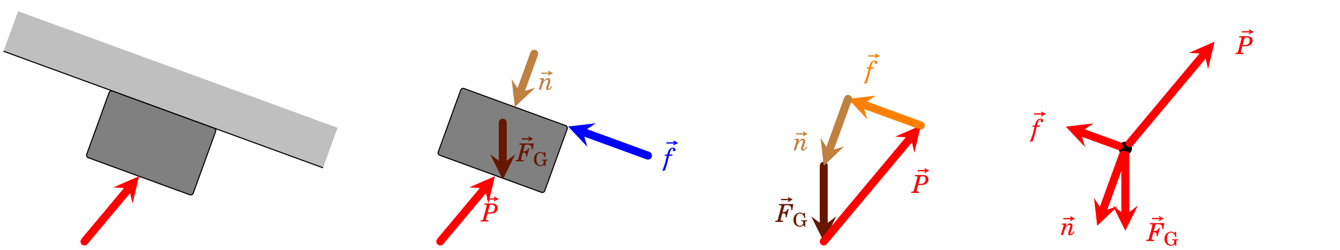
Physically: The applied force, which is larger than the object's weight, is pressing the object into the surface at an angle. The normal points *downwards* (away from the surface) to oppose this. There is a portion of the applied force that is parallel to the surface, which would cause it to slide up the incline, and friction opposes this by pointing *down* the incline.

EXERCISE 1.1.56 The applied force is perpendicular into the surface, and has a magnitude *larger* than the block's weight.



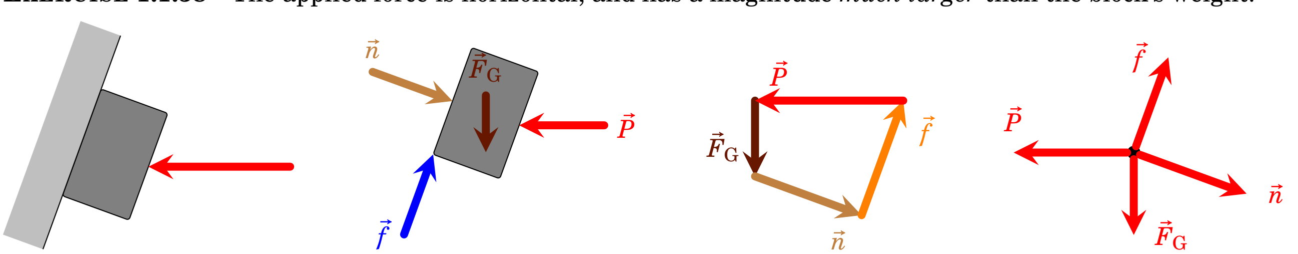
Physically: The applied force, which is larger than the object's weight, is pressing the object directly into the surface. The normal points *downwards* (away from the surface) to oppose this. There is a portion of gravity that is parallel to the surface, which would cause it to slide down the incline, and friction opposes this by pointing *up* the incline.

EXERCISE 1.1.57 The applied force is pointed slightly *down* the incline, and has a magnitude *larger* than the block's weight.



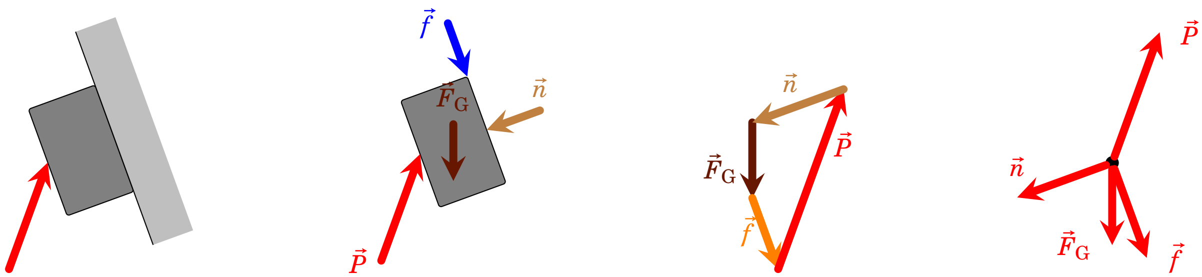
Physically: The applied force, which is larger than the object's weight, is pressing the object into the surface. The normal points *downwards* (away from the surface) to oppose this. Both gravity *and* the push have portions that are parallel to the surface, which would cause it to slide down the incline, and friction opposes this by pointing *up* the incline.

EXERCISE 1.1.58 The applied force is horizontal, and has a magnitude *much larger* than the block's weight.



Physically: The applied force, which is larger than the object's weight, is pressing the object into the surface. The normal points towards the right and slightly downwards (away from the surface) to oppose this. Both gravity *and* the push have portions that are parallel to the surface, which would cause it to slide down the incline, and friction opposes this by pointing *up* the incline.

Exercise 1.1.59 The applied force has a magnitude *larger* than the block's weight.



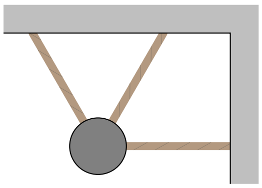
Physically: The applied force, which is larger than the object's weight, has a portion that is pressing the object into the surface. The normal points towards the left and slightly downwards (away from the surface) to oppose this. Both gravity *and* the push have portions that are parallel to the surface, and their sum would cause it to slide down the incline, so friction opposes this by pointing *down* the incline.

1.1.4 Indeterminate Problems

It is possible for there to be more unknown forces than there are are equations. These cases are called *indeterminate*. The possible solutions are subdivided into *cases* categorized by assumed values for one (or more) of the unknowns.

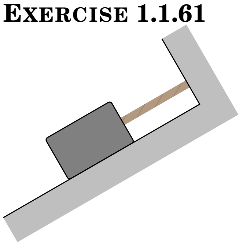
In these exercises there is friction between the ropes and any surfaces that they lay across. This means that, in any cases where a segment of rope lays on a surface, the tension may vary along that length of the rope!

EXERCISE 1.1.60



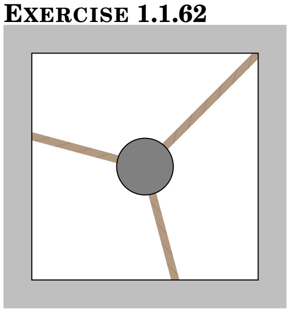
Solve the system (above) for the three unknown tensions in these cases:
(1) The tension in the horizontal rope is zero.
(2) The tension in the rope that points upwards to the right is zero.
(3) The tension in the rope that points upwards to the left equals the object's weight mg.

EXERCISE 1.1.61

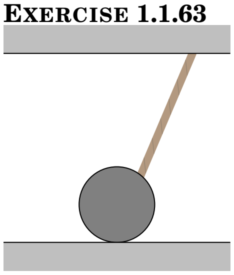


Solve the system (above) for the unknown tension in these cases:
(1) The tension in the rope is very small, but not zero.
(2) The tension in the rope is a value that lets the friction be zero.
(3) The tension in the rope equals the object's weight mg.

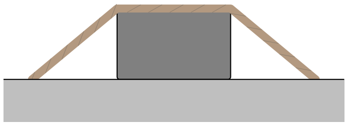
EXERCISE 1.1.62



EXERCISE 1.1.63

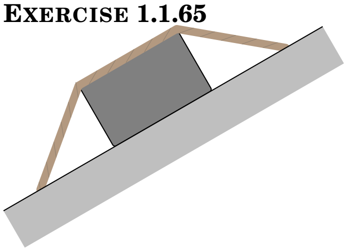


EXERCISE 1.1.64



There is friction between the top of the block and rope. This means that the value of the tension in the rope can be different along its length where lays across the top of the block! Consequently the tension of the segment on the left (from the block to the surface) can be different from the the tension of the segment on the right.

EXERCISE 1.1.65



1.2 Solving problems using *the Process*

The Process

0. Identify the Object!
1. Identify the forces acting on the Object.
2. Draw the Free-Body Diagram.
3. Separately, for each force acting on the Object:
   - draw the coordinates
   - draw the force (vector)
   - determine the components.
4. Use Newton's 1st Law to write the equations to be solved.
5. Solve the equations for the unknown quantities.

Remember to take your time and to work carefully through Steps 0, 1 and 2 – that's where you are doing the physics! Working on these problems is not a race. If you rush through (or skip) those steps, then you will risk getting it *completely wrong*. Working on these problems is where you should practice working methodically through each step of the Process.

1.2.1 Mechanical systems

These Problems are all mechanical (not BIO-mechanical) in that there are no humans or factors of human anatomy involved.

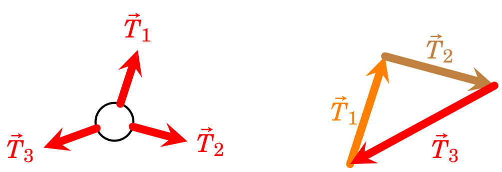
PROBLEM 1.2.01: On the force table a 250 gram mass is hanging at 72° and a 300 gram mass is hanging at 345°. What mass must we hang (and at what direction) to keep the ring at the center in static equilibrium? (The weight of the ring may be ignored since it is much smaller than the tension in the strings will be.)

Solution: We are asked to find a mass and a direction. From our experience in the lab with the force table we can expect that the mass will be something between 200 grams and 600 grams. Also – thinking physically – with the first two strings pulling towards the right and slightly upwards we expect the third string should point towards the left and slightly downwards.

Step 0: The object is the ring at the center of the table.

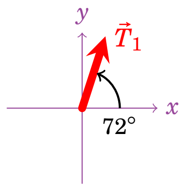
Step 1: The ring is attached to three separate strings, each of which will exert a tension on it. While the ring does have mass, and the gravitational force is not actually zero, its weight will be so much smaller in magnitude than the tensions acting on it that we can neglect the effect of its weight.

Step 2: The Free-Body Diagram and check of sum of forces:

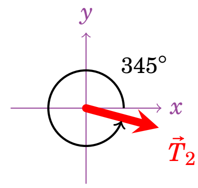


From these qualitative constructions we can see that the third string should point towards the left and slightly downwards, into the third quadrant.

Step 3: The components of the known and unknown forces:



$T_{1,x} = (0.250\,\mathrm{kg})(9.81\,\mathrm{N/kg})\cos 72° = +0.758\,\mathrm{N}$
$T_{1,y} = (0.250\,\mathrm{kg})(9.81\,\mathrm{N/kg})\sin 72° = +2.332\,\mathrm{N}$



$T_{2,x} = (0.300\,\mathrm{kg})(9.81\,\mathrm{N/kg})\cos 345° = +2.843\,\mathrm{N}$
$T_{2,y} = (0.300\,\mathrm{kg})(9.81\,\mathrm{N/kg})\sin 345° = -0.762\,\mathrm{N}$

The third tension is completely unknown. We will solve for the components, from which we can get the magnitude and direction. From our FBD and check-of-sum of forces we expect that this force will point into the third quadrant.

Step 4: Newton's 1st Law applies to the ring at the center of the table:

$$\sum \vec{F} = \vec{0}\,\mathrm{N} \tag{1.1}$$

In this specific situation the sum of forces is

$$\vec{T}_1 + \vec{T}_2 + \vec{T}_3 = \vec{0}\,\mathrm{N} \tag{1.2}$$

The x-component of that equation is:

$$T_{1,x} + T_{2,x} + T_{3,x} = 0\,\mathrm{N} \tag{1.3}$$
$$(+0.758\,\mathrm{N}) + (+2.843\,\mathrm{N}) + T_{3,x} = 0\,\mathrm{N} \tag{1.4}$$

The y-component of that equation is:

$$T_{1,y} + T_{2,y} + T_{3,y} = 0\,\mathrm{N} \tag{1.5}$$
$$(+2.332\,\mathrm{N}) + (-0.762\,\mathrm{N}) + T_{3,y} = 0\,\mathrm{N} \tag{1.6}$$

Step 5: Solving the equations we had above gives

$$T_{3,x} = -3.601\,\mathrm{N} \tag{1.7}$$
$$T_{3,y} = -1.571\,\mathrm{N} \tag{1.8}$$

From these components we find that the magnitude is

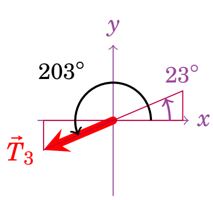
$$T_3 = \sqrt{(T_{3,x})^2 + (T_{3,y})^2} = \sqrt{(-3.601\,\mathrm{N})^2 + (-1.571\,\mathrm{N})^2} = 3.928\,\mathrm{N} \tag{1.9}$$

From this magnitude of tension we can determine the mass hanging from the end of the string: $m_3 = T_3/g = 3.928\,\mathrm{N}/9.81\,\mathrm{N/kg} = 0.400\,\mathrm{kg}$.

The components tell us that the normal points into the third quadrant (that the angle θ in the diagram is between 180° and 270°). Using the inverse tangent function, we obtain

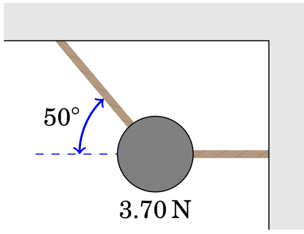
$$\tan^{-1}(T_{3,y}/T_{3,x}) = \tan^{-1}\left((-1.571\,\mathrm{N})/(-3.601\,\mathrm{N})\right) = 23.57° \tag{1.10}$$

The calculator's answer is in the first quadrant, formed by a triangle whose x and y sides are both positive. We translate this result into the correct quadrant by adding 180°:



Answer: Thus, the mass hanging from, and the direction of, the third string are $m_3 = 0.400$ kg = 400 grams and $\theta = 203°$. Both of these results meet our expectations.

PROBLEM 1.2.02: An object of weight 3.70 N is hanging by two ropes, as shown. Find the magnitude of the tension in each rope.

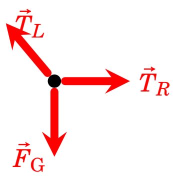


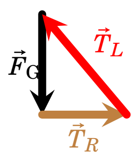
Solution:

Step 0: The Object is the sphere.

Step 1: Gravity pulls downwards on the object (the weight of 3.70 N). Each rope will pull on the object. The object is not touching a surface, so there will not be any normal nor any friction.

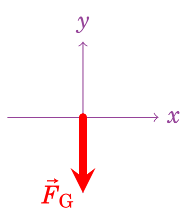
Step 2: The Free-Body Diagram and check of sum of forces:



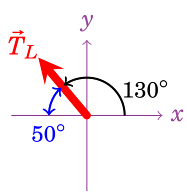


From these qualitative constructions we can see that the weight is supported by the vertical component of the tension on the left, and the tension towards the right is countered by the horizontal component of the tension on the left. For these reasons we can expect that the magnitude of the tension in the rope on the left will be greater than the weight ($T_L > mg$) and greater than in the rope on the right ($T_L > T_R$).

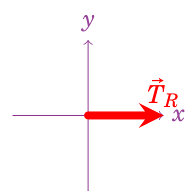
Step 3: The components of the known and unknown forces:



$F_{G,x} = 0\,\text{N}$
$F_{G,y} = -3.70\,\text{N}$



$T_{L,x} = T_L \cos 130°$
$T_{L,y} = T_L \sin 130°$



$T_{R,x} = +T_R$
$T_{R,y} = 0\,\text{N}$

The magnitudes of both tensions are unknowns that we must solve for.

Step 4: Newton's 1st Law applied to the Object is

$$\vec{F}_G + \vec{T}_L + \vec{T}_R = \vec{0}\,\text{N} \tag{1.11}$$

The x-component of this equation is

$$F_{G,x} + T_{L,x} + T_{R,x} = 0\,\text{N} \tag{1.12}$$
$$(0\,\text{N}) + (T_L \cos 130°) + (+T_R) = 0\,\text{N} \tag{1.13}$$

The y-component of this equation is

$$F_{G,y} + T_{L,y} + T_{R,y} = 0\,\text{N} \tag{1.14}$$
$$(-3.70\,\text{N}) + (T_L \sin 130°) + (0\,\text{N}) = 0\,\text{N} \tag{1.15}$$

Step 5: In the y-component equation we can see that there is only one unknown, which can be easily solved for (if we are careful with the signs!):

$$(-3.70\,\text{N}) + (T_L \sin 130°) + (0\,\text{N}) = 0\,\text{N} \tag{1.16}$$
$$T_L = -\frac{(-3.70\,\text{N})}{\sin 130°} = +4.83\,\text{N} \tag{1.17}$$

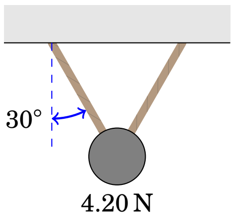
This gets substituted into the x-component equation, which then solves the other unknown:

$$(0\,\mathrm{N}) + (T_L \cos 130^\circ) + (+T_R) = 0\,\mathrm{N} \tag{1.18}$$

$$T_R = -T_L \cos 130^\circ = -(+4.83\,\mathrm{N}) \cos 130^\circ = +3.11\,\mathrm{N} \tag{1.19}$$

Answer: We found that $T_L = 4.83$ N and that $T_R = 3.11$ N. This meets our expectations that $T_L > mg$ and that $T_L > T_R$.

PROBLEM 1.2.03: An object of weight 4.20 N is hanging from two ropes that are symmetric (as shown). Find the magnitude of the tension in each rope.

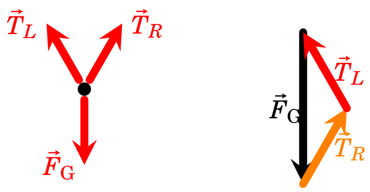


Solution: Since the ropes are attached symmetrically we can expect that the two tensions will have the same magnitude.

Step 0: The Object is the sphere.

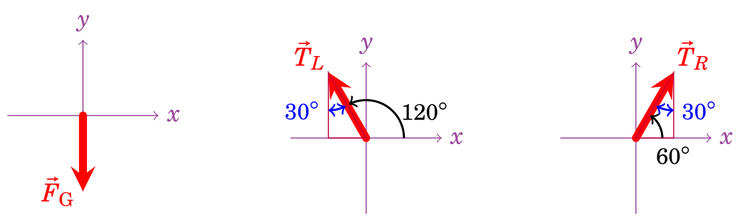
Step 1: Gravity pulls downwards on the object (the weight of 4.20 N). Each rope will pull on the object. The object is not touching a surface so there will not be a normal, nor any friction.

Step 2: The Free-Body Diagram and check of sum of forces:



From these qualitative constructions we can see that the tensions should have the same magnitude $T_L = T_R$. (Be careful and note that the forces are along different directions, so that the *vectors* are not equal $\vec{T}_L \neq \vec{T}_R$.) Each rope will support half the weight, but will also have to counter the horizontal component of tension in the other rope. So can expect that each tension will have a greater magnitude than half the weight.

Step 3: The components of the known and unknown forces:



$F_{G,x} = 0\,\text{N}$ $\quad T_{L,x} = T_L \cos 120°$ $\quad T_{R,x} = T_R \cos 60°$

$F_{G,y} = -4.20\,\text{N}$ $\quad T_{L,y} = T_L \sin 120°$ $\quad T_{R,y} = T_R \sin 60°$

The magnitudes of both tensions are unknowns that we must solve for.

Step 4: Newton's 1st Law applied to the Object is

$$\vec{F}_G + \vec{T}_L + \vec{T}_R = \vec{0}\,\text{N} \tag{1.20}$$

The x-component of this equation is

$$F_{G,x} + T_{L,x} + T_{R,x} = 0\,\text{N} \tag{1.21}$$
$$(0\,\text{N}) + (T_L \cos 120°) + (+T_R \cos 60°) = 0\,\text{N} \tag{1.22}$$

The y-component of this equation is

$$F_{G,y} + T_{L,y} + T_{R,y} = 0\,\text{N} \tag{1.23}$$
$$(-4.20\,\text{N}) + (T_L \sin 120°) + (T_R \sin 60°) = 0\,\text{N} \tag{1.24}$$

Step 5: We see that both unknowns appear in each equation. There is no single equation where we can get an unknown by itself. To solve we need to first express one unknown in terms of the other, then substitute that into the other equation.

From the x-component equation we can obtain

$$(0\,\mathrm{N}) + (T_L \cos 120^\circ) + (+T_R \cos 60^\circ) = 0\,\mathrm{N} \tag{1.25}$$

$$T_L = -\frac{\cos 60^\circ}{\cos 120^\circ} T_R = +T_R \tag{1.26}$$

(This shows that the two tension have equal magnitudes, as we expected from the symmetry of the system.) That fact can then be substituted in the y-component equation: The y-component of this equation is

$$(-4.20\,\mathrm{N}) + (T_L \sin 120^\circ) + (T_R \sin 60^\circ) = 0\,\mathrm{N} \tag{1.27}$$

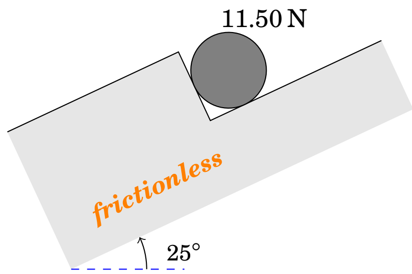
$$(-4.20\,\mathrm{N}) + (T_L \sin 120^\circ) + (T_L \sin 60^\circ) = 0\,\mathrm{N} \tag{1.28}$$

$$T_L (\sin 120^\circ + \sin 60^\circ) = +4.20\,\mathrm{N} \tag{1.29}$$

$$T_L = +2.43\,\mathrm{N} \tag{1.30}$$

Answer: Both tension have a magnitude of 2.43 N. (As expected the tensions were equal in magnitude, and greater than half the weight.)

PROBLEM 1.2.04: A spherical object of weight 11.50 N rests in a corner of a frictionless surface. Find the magnitude of the normal exerted by each surface.

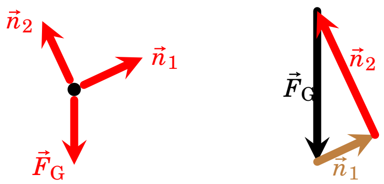


Solution:

Step 0: The Object is the sphere.

Step 1: Gravity pulls downwards on the object (the weight of 11.50 N). Each part of the surface in contact with the object will exert a contact force. The surface is frictionless, so these contact forces will be just the normals. There are no ropes, nor any other external forces acting on the object.

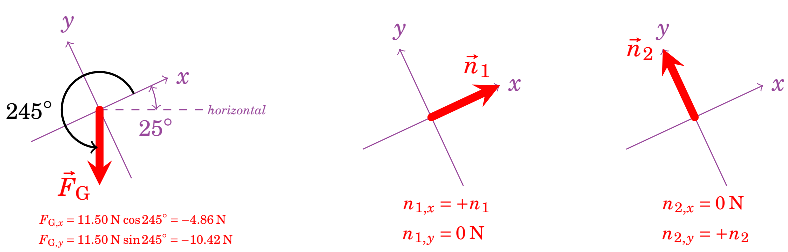
Step 2: The Free-Body Diagram and check of sum of forces:



From these qualitative constructions we can see that the two normals are just opposites to the components of the weight, as measured relative to the incline. This encourages us to use a coordinate system that is tilted along the surface when finding the components in the next step.

Another feature we notice is that, with the incline at only 25°, the part of the surface that is close to horizontal will support more of the weight, and should have a larger magnitude than the other, more vertical, surface.

Step 3: The components of the known and unknown forces:



Here we can see that the choice of coordinates will make the algebra very simple, with only one unknown along each of the axes.

Step 4: Newton's 1st Law applied to the Object is

$$\vec{F}_G + \vec{n}_1 + \vec{n}_2 = \vec{0}\,\mathrm{N} \tag{1.31}$$

The x-component of this equation is

$$F_{G,x} + n_{1,x} + n_{2,x} = 0\,\mathrm{N} \tag{1.32}$$
$$(-4.86\,\mathrm{N}) + (+n_1) + (0\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.33}$$

The y-component of this equation is

$$F_{G,x} + n_{1,x} + n_{2,x} = 0\,\mathrm{N} \tag{1.34}$$
$$(-10.42\,\mathrm{N}) + (0\,\mathrm{N}) + (+n_2) = 0\,\mathrm{N} \tag{1.35}$$

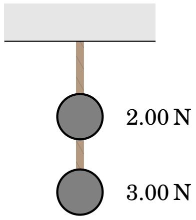
Step 5: The tricky part of this problem was the geometry. Here, the algebra is trivial, and we can read the answers directly from the equations above:

$$n_1 = +4.86\,\mathrm{N} \tag{1.36}$$

$$n_2 = +10.42\,\mathrm{N}. \tag{1.37}$$

Answer: The portion of the surface that is most horizontal exerts a normal of magnitude 10.42 N, while the portion that is mostly vertical exerts a normal of magnitude 4.86 N. (As expected from our qualitative construction in the check of the sum of forces, the more horizontal surface exerts a larger normal, and supports most of the object's weight.)

PROBLEM 1.2.05: Two objects are hanging as shown. Find the magnitude of the tension in each rope.



Solution:

Step 0: There are *two* objects in this system! The Process applies to a single object, so we will have to do the Process once for one object, and then, *separately*, do the Process for the other object. Because the two objects are attached to each other they are *interacting*. For this reason the results we obtain for one object may be required to solve what is happening to the other object. In contexts like this it will Newton's 3rd Law that relates these pairs of interactions together.

For the purposes of this problem we will call the object above "object A" and the object below "object B".

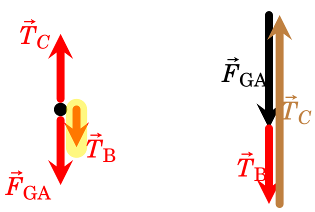
Step 1: Interactions must be listed for each object, separately.

For object A, there is the interaction with Earth (gravity), the rope above it (which connects it to the ceiling), and the rope below it (through with it interacts with object B).

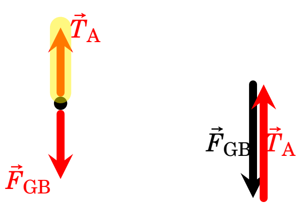
For object B, there is the interaction with Earth (gravity), and the rope above it (through with it interacts with object A).

Step 2: When we construct the Free-Body Diagram and check of sum of forces we must remember that Newton's 3rd Law relates the force that A exerts on B to the force that B exerts on A.

For object A we will name $\vec{T}_{\mathrm{B}}$ the force that object B exerts on it through the connecting rope. The force that the ceiling exerts through the rope above object A we will call $\vec{T}_C$.

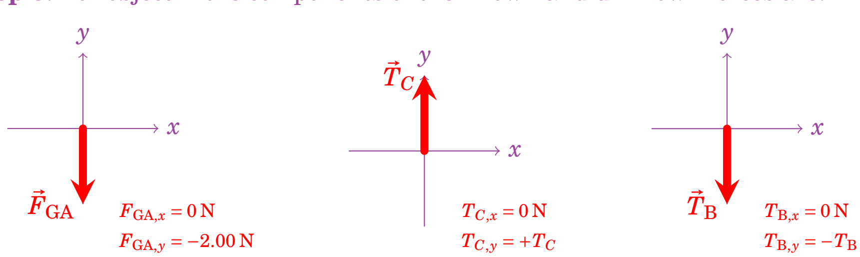


For object B we will name $\vec{T}_{\mathrm{A}}$ the force that object A exerts on it through the connecting rope.

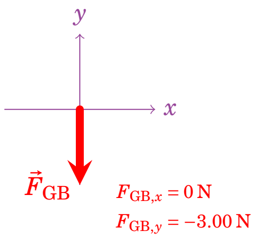


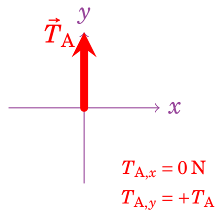
In the Free-Body Diagrams above the forces of interaction that are related by Newton's 3rd Law have been high-lighted in yellow. The force that B exerts on A ($\vec{T}_{\mathrm{B}}$) is equal in magnitude, but opposite in direction, to the the force that A exerts on B ($\vec{T}_{\mathrm{A}}$).

Step 3: For object A the components of the known and unknown forces are:



For object B the components of the known and unknown forces are:





We we are solving for the unknowns we must remember that Newton's 3rd Law will require that the magnitudes T_{A} and T_{B} equal each other ($T_{\mathrm{B}} = T_{\mathrm{A}}$). The directions being opposite ($\vec{T}_{\mathrm{B}} = -\vec{T}_{\mathrm{A}}$) we have already accounted for in the signs of the components, above.

Step 4: Newton's 1st Law applied to object A gives

$$\vec{F}_{\mathrm{GA}} + \vec{T}_C + \vec{T}_{\mathrm{B}} = \vec{0}\,\mathrm{N} \tag{1.38}$$

and applied to object B gives

$$\vec{F}_{\mathrm{GB}} + \vec{T}_{\mathrm{A}} = \vec{0}\,\mathrm{N} \tag{1.39}$$

(Read these equations carefully, paying close attention to the sub-scripts.) In each of those equations the x-components of every vector is zero. Consequently only the y-components of those equations are relevant. The y-component of Newton's 1st Law applied to object A is

$$F_{\mathrm{GA},y} + T_{C,x} + T_{\mathrm{B},y} = 0\,\mathrm{N} \tag{1.40}$$
$$-2.00\,\mathrm{N} + T_C - T_{\mathrm{B}} = 0\,\mathrm{N} \tag{1.41}$$

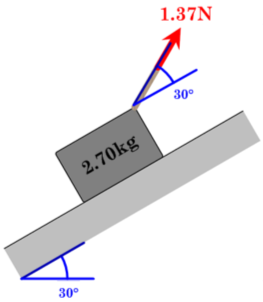
while the y-component of Newton's 1st Law applied to object B is

$$F_{\mathrm{GB},y} + T_{\mathrm{A},y} = 0\,\mathrm{N} \tag{1.42}$$
$$-3.00\,\mathrm{N} + T_{\mathrm{A}} = 0\,\mathrm{N} \tag{1.43}$$

Step 5: Solving the y-component equation for object B tells us that $T_{\mathrm{A}} = 3.00\,\mathrm{N}$. This is the magnitude of the tension in the segment of rope between the two objects. Recalling that Newton's 3rd Law requires $T_{\mathrm{B}} = T_{\mathrm{A}}$, we obtain $T_{\mathrm{B}} = 3.00\,\mathrm{N}$. Using this fact we can solve the y-component equation for object A to obtain $T_C = 5.00\,\mathrm{N}$, which is the magnitude of the tension in the segment of rope between object A and the ceiling.

Answer: The rope above the objects supports the weight of both objects with a tension of magnitude 5.00 N. The rope between the objects supports the weight of the lower object with a tension of magnitude 3.00 N.

Problem 1.2.06: An object remains at rest on an incline when a force is applied as shown in the diagram. The mass of the object is 2.70 kg and the magnitude of the applied pull is 1.37 N. What is the magnitude of the normal acting on the object? What is the magnitude and direction of the static friction acting on the object?

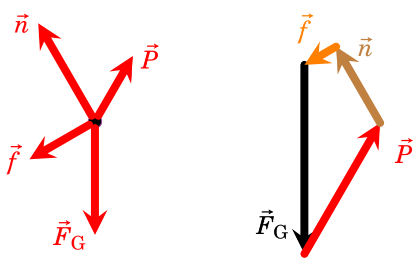


Solution:

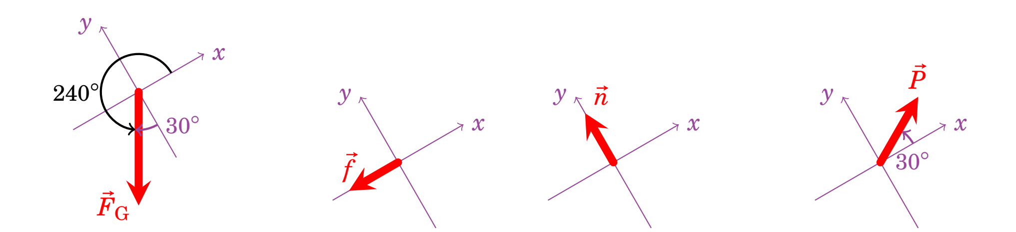
Step 0: The rectangular Object is given to us.

Step 1: The object is interacting with the Earth (gravity), the surface (normal and friction), and is being pulled. So there will be four distinct forces acting on the object. Since the applied pull is pointed up the incline we will *guess* that the friction points down the incline.

Step 2: The Free-Body Diagram and check of sum of forces:



Step 3: The components of the known and unknown forces are:



$F_{\mathrm{G},x} = mg\cos 240^\circ = -13.244\,\mathrm{N}$
$F_{\mathrm{G},y} = mg\sin 240^\circ = -22.938\,\mathrm{N}$

$f_x = -f$
$f_y = 0\,\mathrm{N}$

$n_x = 0\,\mathrm{N}$
$n_y = +n$

$P_x = 1.37\,\mathrm{N}\cos 30^\circ = +1.186\,\mathrm{N}$
$P_y = 1.37\,\mathrm{N}\sin 30^\circ = +0.685\,\mathrm{N}$

The magnitudes of the normal (n) and friction (f) are unknowns for which we must solve. If we are incorrect in our guess about the direction of friction, then we will obtain a negative value for "f". We choose a tilted coordinate system since it simplifies the algebra for the unknowns by not mixing each unknown onto both axes.

Step 4: In this situation Newton's 1st Law applies to the Object:

$$\sum \vec{F} = \vec{0}\,\mathrm{N} \tag{1.44}$$

In this specific situation the sum of forces is

$$\vec{F}_{\mathrm{G}} + \vec{f} + \vec{n} + \vec{P} = \vec{0}\,\mathrm{N} \tag{1.45}$$

The components of this constraint are the equations we must solve. The x-component of this equation is:

$$F_{\mathrm{G},x} + f_x + n_x + P_x = 0\,\mathrm{N} \tag{1.46}$$
$$(-13.244\,\mathrm{N}) + (-f) + (0\,\mathrm{N}) + (+1.186\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.47}$$

The y-component of this equation is:

$$F_{\mathrm{G},y} + f_y + n_y + P_y = 0\,\mathrm{N} \tag{1.48}$$
$$(-22.253\,\mathrm{N}) + (0\,\mathrm{N}) + (+n) + (+0.685\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.49}$$

Step 5: The x-component equation can be solved for the magnitude of the friction:

$$(-13.244\,\mathrm{N}) + (-f) + (0\,\mathrm{N}) + (+1.186\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.50}$$

$$f = -12.057\,\mathrm{N} \tag{1.51}$$

Since a magnitude must be positive, this signaled that our assumption about the direction of friction was incorrect. To maintain static equilibrium, friction must point ***upwards*** along the incline.

The y-component equation can be solved for the magnitude of the normal:

$$(-22.253\,\mathrm{N}) + (0\,\mathrm{N}) + (+n) + (+0.685\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.52}$$

$$n = +22.253\,\mathrm{N} \tag{1.53}$$

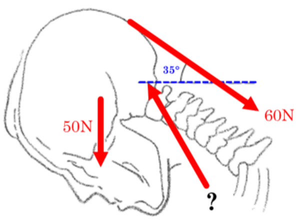
As expect from the Free-Body Diagram, the normal is less than the weight since the pull acts slightly away from the surface.

Answer: The normal acting on the object is 22.3 N. The force of static friction acting on the object is 12.1 N pointing *upwards* along the incline.

1.2.2 Biomechanical systems

These Problems are all biomechanical in that there are humans or factors of human anatomy involved.

PROBLEM 1.2.07: A student looking at their cellphone has their head bent over, as shown. The weight of their head is 50N, and the tension in the muscles attached to the base of their skull is 60 N. What force is exerted by the 1st cervical vertebrae onto the base of their skull? (Specify magnitude and direction.)

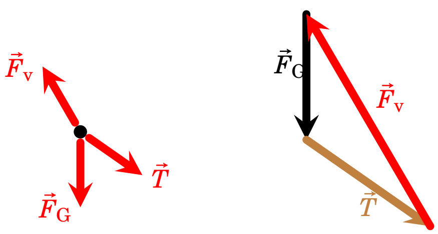


Solution:

Step 0: The Object is the student's head.

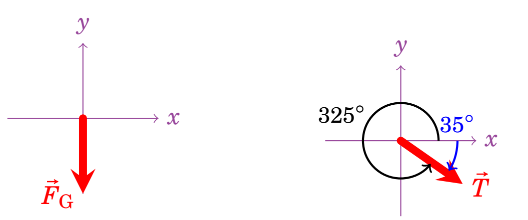
Step 1: Gravity pulls downwards on the student's head (the 50 N force). The muscles on the back of their neck pull (the 60 N force) their head towards the vertebrae, which produces a reaction (the unknown force).

Step 2: The Free-Body Diagram and check of sum of forces:



From these qualitative constructions we can see that the force exerted by the vertebrae ($\vec{F}_{\mathrm{v}}$) should be upwards and towards the left. Also it should be larger in magnitude than either of the other forces since those forces both contribute in the same general direction (downwards).

Step 3: The components of the known and unknown forces:



$F_{\mathrm{G},x} = 0\,\mathrm{N}$

$F_{\mathrm{G},y} = -50\,\mathrm{N}$

$T_x = 60\,\mathrm{N}\cos 325° = +49.15\,\mathrm{N}$

$T_y = 60\,\mathrm{N}\sin 325° = -34.42\,\mathrm{N}$

Even though we expect that the force of the vertebrae on the head will be upwards and towards the left we will pretend total ignorance of this force's magnitude and direction. Both components, $F_{\mathrm{v},x}$ and $F_{\mathrm{v},x}$, are unknowns to be solved for.

Step 4: Newton's 1st Law applies to the student's head:

$$\sum \vec{F} = \vec{0}\,\mathrm{N} \tag{1.54}$$

In this specific situation the sum of forces is

$$\vec{F}_{\mathrm{G}} + \vec{T} + \vec{F}_{\mathrm{v}} = \vec{0}\,\mathrm{N} \tag{1.55}$$

The x-component of that equation is:

$$F_{\mathrm{G},x} + T_x + F_{\mathrm{v},x} = 0\,\mathrm{N} \tag{1.56}$$

$$(0\,\mathrm{N}) + (+49.15\,\mathrm{N}) + F_{\mathrm{v},x} = 0\,\mathrm{N} \tag{1.57}$$

The y-component of that equation is:

$$F_{G,y} + T_y + F_{v,y} = 0\,\mathrm{N} \tag{1.58}$$

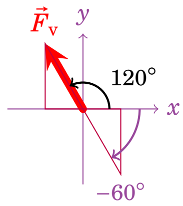
$$(-50\,\mathrm{N}) + (-34.42\,\mathrm{N}) + F_{v,y} = 0\,\mathrm{N} \tag{1.59}$$

Step 5: Solving those equations above gives $F_{v,x} = -49.15\,\mathrm{N}$ and $F_{v,y} = +84.42\,\mathrm{N}$.

The components tell us that this force points into the second quadrant (that the angle θ in the diagram is between 90° and 180°). Using the inverse tangent function, we obtain

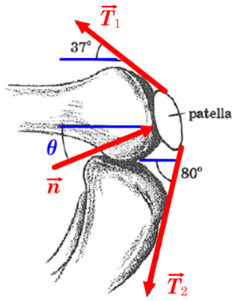
$$\tan^{-1}(F_{v,y}/F_{v,x}) = \tan^{-1}\big((+84.42\,\mathrm{N})/(-49.15\,\mathrm{N})\big) = -59.79° \approx -60° \tag{1.60}$$

The calculator's answer is in the fourth quadrant, formed by a triangle whose x side is positive and whose y side is negative. We translate this result into the correct quadrant by adding 180°:



Answer: The magnitude of the force of the vertebrae on the head is 97.68 N, and the direction (angle measured counter-clockwise from the $+x$-axis) is 120°. This meets both of the expectations we generated from constructing the check of the sum of forces.

PROBLEM 1.2.08: The tendon from the quadraceps muscle (thigh) passes over the patella (kneecap) to attach to the tibia (shin bone). The tension in the tendon is 1333N. What are the magnitude and direction of the contact force between the patella and the femur (thigh bone)? (There is essentially no friction between the patella and femur. The mass of the patella is only a few grams, so gravity may be ignored relative to the tension in the tendon.)



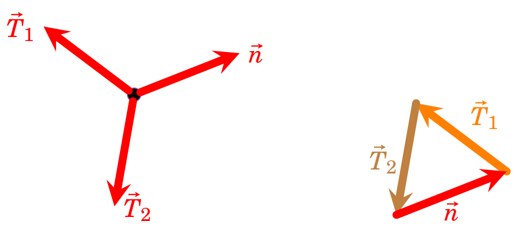
Solution:

Step 0: The Object is the patella (the kneecap).

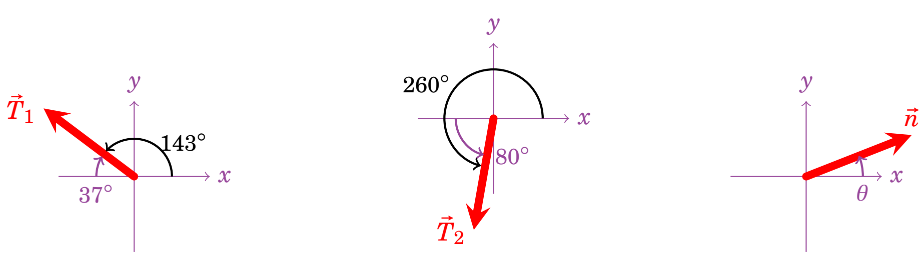
Step 1: The patella is attached to a tendon both above and below, so there will be two tensions. These tensions press the patella against the end of the femur, so there will be a normal force, and we are told there is no friction.

The patella is a small piece of bone with a mass on the order of 10 grams. Its weight (roughly 0.1 N) is completely negligible in comparison to the tensions and contact forces exerted on it, so we can forget about gravity.

Step 2: The Free-Body Diagram and check of sum of forces:



Step 3: The components of the known and unknown forces:



$T_{1,x} = 1333\,\text{N}\cos 143^\circ = -1065\,\text{N}$ $\quad$ $T_{2,x} = 1333\,\text{N}\cos 260^\circ = -231\,\text{N}$ $\quad$ $n_x = n_x$

$T_{1,y} = 1333\,\text{N}\sin 143^\circ = +802\,\text{N}$ $\quad$ $T_{2,y} = 1333\,\text{N}\sin 260^\circ = -1313\,\text{N}$ $\quad$ $n_y = n_y$

The components of the normal, n_x and n_y, are unknowns for which we must solve. From their values we can calculate the magnitude and direction, which we are asked for.

Step 4: This is a situation Newton's 1st Law applies to the Object:

$$\sum \vec{F} = \vec{0}\,\text{N} \tag{1.61}$$

In this specific situation the sum of forces is

$$\vec{T}_1 + \vec{T}_2 + \vec{n} = \vec{0}\,\text{N} \tag{1.62}$$

The components of this constraint are the equations we must solve. The x-component is:

$$T_{1,x} + T_{2,x} + n_x = 0\,\text{N} \tag{1.63}$$
$$(-1065\,\text{N}) + (-231\,\text{N}) + n_x = 0\,\text{N} \tag{1.64}$$

The y-component is:

$$T_{1,y} + T_{2,y} + n_y = 0\,\mathrm{N} \tag{1.65}$$

$$(+802\,\mathrm{N}) + (-1313\,\mathrm{N}) + n_y = 0\,\mathrm{N} \tag{1.66}$$

Step 5: Solving the equations above, we get

$$n_x = +1296\,\mathrm{N} \tag{1.67}$$

$$n_y = +511\,\mathrm{N} \tag{1.68}$$

From these components we find that the magnitude is

$$n = \sqrt{n_x^2 + n_y^2} = \sqrt{(+1296\,\mathrm{N})^2 + (+511\,\mathrm{N})^2} = 1393\,\mathrm{N} \tag{1.69}$$

The components tell us that the normal points into the first quadrant (that the angle θ in the diagram is between 0° and 90°). Using the inverse tangent function, we obtain

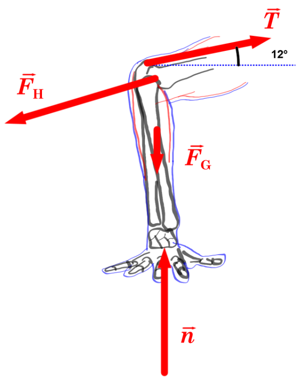
$$\tan^{-1}(n_y/n_x) = \tan^{-1}\big((+511\,\mathrm{N})/(+1296\,\mathrm{N})\big) = 21.5^\circ \tag{1.70}$$

This is in the first quadrant, as required.

Answer: Thus, the normal force of the end of the femur acting on the patella has a magnitude $n = 1393\,\mathrm{N} \approx 1.4\,\mathrm{kN}$, and points away from the femur at $\theta = 21.5^\circ$ above the horizontal.

Problem 1.2.09: A person is doing a push-up. Choosing the right forearm (including the hand) as the Object, we want to find the force acting at the elbow joint where the bones of the upper and lower arm meet. There is the contact force $\vec{n}$ with the floor (magnitude 200.0 N) that acts vertically upwards. There is, of course, the weight $\vec{F}_G$ of the forearm itself (86.8 N), acting vertically downwards. The tension $\vec{T}$ in the triceps muscle, attached to the ulna, exerts a force of 1 826.6 N along a direction of 12° above the horizontal.

Find the magnitude and the direction of the force $\vec{F}_H$ exerted by the humerus bone (of the upper arm) onto the ulna bone (of the forearm).



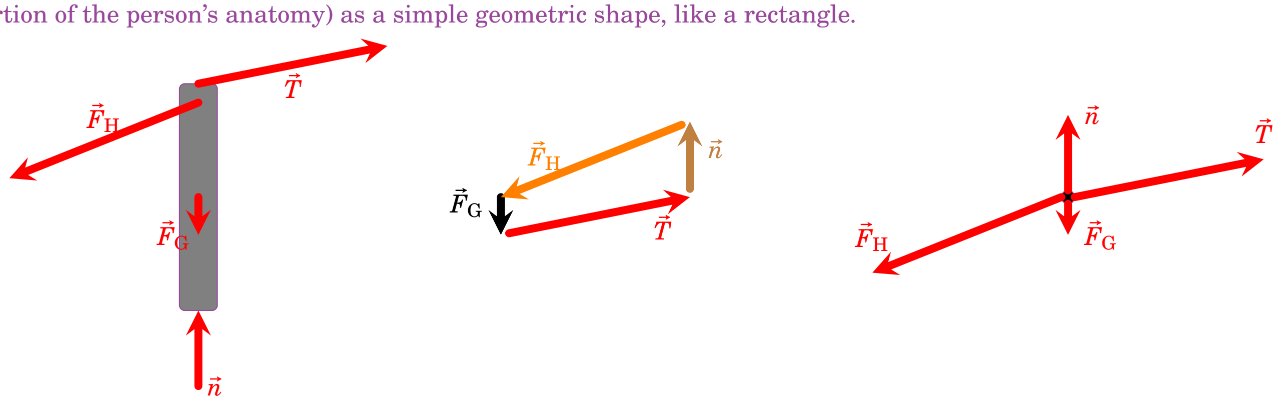
Solution:

Step 0: We are told in the statement of the problem to take the forearm as the object.

Step 1: We are told what forces are acting on the object explicitly in the statement of the problem. But, for the sake of practice, it worthwhile pausing to name the things the object is interacting with.

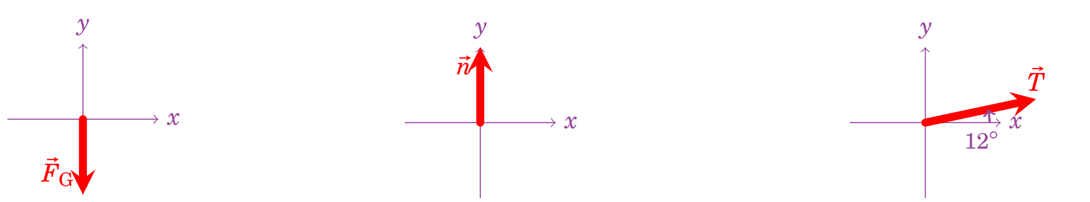
The person's forearm is interacting with the Earth, the floor (that they're doing push-ups on), and *their upper arm*. In the biomechanical context, when we study the forces acting on a *portion* of a person's anatomy, we must always consider the forces in the connective tissues and the forces of contact between the bones.

Step 2: The Free-Body Diagram and the check of sum of forces:
When drawing the FBD biomechanical systems, you do not have to make an anatomically correct illustration. A "cartoon" will do. The purpose is to sketch the forces acting on the object. To that end, we can sketch the object (the portion of the person's anatomy) as a simple geometric shape, like a rectangle.



The normal is larger than the weight of the object (the person's forearm) because it is supporting not just the object, but a portion of the person's entire weight. That additional weight is being transferred to the forearm by the forces acting at the elbow.

Step 3: The components of the known and unknown forces:



$F_{G,x} = 0\,\text{N}$
$F_{G,y} = -86.8\,\text{N}$

$n_x = 0\,\text{N}$
$n_y = +200.0\,\text{N}$

$T_x = 1826.6\,\text{N}\cos 12° = +1786.7\,\text{N}$
$T_y = 1826.6\,\text{N}\sin 12° = +379.8\,\text{N}$

The components of $\vec{F}_{\text{H}}$, the contact force at the elbow between the humerus and the ulna, are completely unknown. From the qualitative check of the sum of forces in Step 2 we anticipate that this force points into the third quadrant. But quantitatively, we will leave $F_{\text{H},x}$ and $F_{\text{H},y}$ as separate unknowns.

Step 4: Newton's 1st Law applies to the person's forearm:

$$\vec{F}_{\text{G}} + \vec{N} + \vec{T} + \vec{F}_{\text{H}} = \vec{0}\,\text{N} \tag{1.71}$$

The x-component of this equation is:

$$F_{\text{G},x} + n_x + T_x + F_{\text{H},x} = 0\,\text{N} \tag{1.72}$$

$$0\,\text{N} + 0\,\text{N} + (+1786.7\,\text{N}) + F_{\text{H},x} = 0\,\text{N} \tag{1.73}$$

The y-component of this equation is:

$$F_{\text{G},y} + n_y + T_y + F_{\text{H},y} = 0\,\text{N} \tag{1.74}$$

$$(-86.8\,\text{N}) + (+200.0\,\text{N}) + (+379.8\,\text{N}) + F_{\text{H},y} = 0\,\text{N} \tag{1.75}$$

Step 5: Solving the x-component equation

$$0\,\text{N} + 0\,\text{N} + (+1786.7\,\text{N}) + F_{\text{H},x} = 0\,\text{N} \tag{1.76}$$

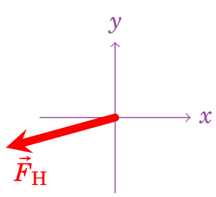
$$F_{\text{H},x} = -1786.7\,\text{N} \tag{1.77}$$

Solving the y-component equation

$$(-86.8\,\text{N}) + (+200.0\,\text{N}) + (+379.8\,\text{N}) + F_{\text{H},y} = 0\,\text{N} \tag{1.78}$$

$$F_{\text{H},y} = -493.0\,\text{N} \tag{1.79}$$

From the components, which are both negative, we see that $\vec{F}_{\text{H}}$ points into the third quadrant.



We were asked to find the magnitude and direction of $\vec{F}_{\text{H}}$. From the components, the magnitude is

$$F_{\text{H}} = \sqrt{(F_{\text{H},x})^2 + (F_{\text{H},y})^2} = \sqrt{(-1786.7\,\text{N})^2 + (-493.0\,\text{N})^2} = 1853.5\,\text{N} \tag{1.80}$$

If we use the inverse tangent function, our calculator tells us

$$\theta = \tan^{-1}\left(F_{\text{H},y}/F_{\text{H},x}\right) = \tan^{-1}\left(\frac{-493.0\,\text{N}}{-1786.7\,\text{N}}\right) = \tan^{-1}(0.2759) = +15.4^\circ \tag{1.81}$$

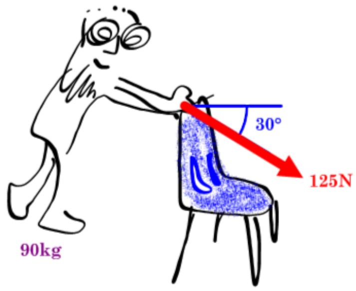
which is in the first quadrant. But we know that the vector points into the third quadrant. Thus the correct angle is 15.4° below the horizontal towards the left

$$\theta = +15.4^\circ + 180^\circ = +195^\circ \tag{1.82}$$

which is equivalent to -165°.

Answer: The force of contact of the humerus bone on the ulna bone at the elbow has a magnitude of 1853.5 N and points 15.4° below the horizontal towards the left.

PROBLEM 1.2.10: As I push a chair away from me (as shown below) what are the normal and friction forces at my feet if I do not slide while pushing?



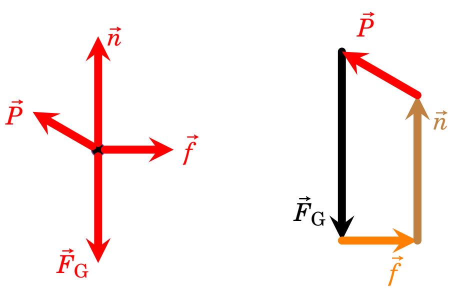
Solution:

Step 0: The problem is asking about the forces acting on me, from which we can conclude that *I* am the object, not the chair.

Step 1: The force that we are told about is the force that I am exerting on the chair. But the question is about the forces acting on me. So we need to know the force that the chair exerts on me. By Newton's 3rd Law, the chair exerts a force that is equal in magnitude (125 N) and opposite in direction (30° above the horizontal, towards the left). If the chair is pushing me towards the left, then friction will have to point towards the right.

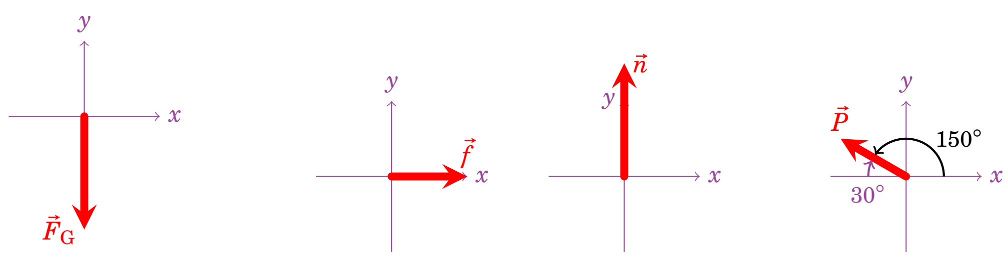
Present as always will also be gravity (due to my interaction with the Earth) and the normal force of the floor upon my feet.

Step 2: The Free-Body Diagram and the check of sum of forces:



Notice how my pushing slightly downwards on the chair has it pushing slight upwards on me. For this reason the normal is slightly reduced in magnitude to something less than my weight.

Step 3: The components of the known and unknown forces:



$F_{\mathrm{G},x} = 0\,\mathrm{N}$
$F_{\mathrm{G},y} = -(90\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = -882.9\,\mathrm{N}$

$f_x = +f$
$f_y = 0\,\mathrm{N}$

$n_x = 0\,\mathrm{N}$
$n_y = +n$

$P_x = 125\,\mathrm{N}\cos 150^\circ = -108.3\,\mathrm{N}$
$P_y = 125\,\mathrm{N}\sin 150^\circ = +62.5\,\mathrm{N}$

The magnitudes of the normal (n) and friction (f) are unknowns for which we must solve.

Step 4: In this situation Newton's 1st Law applies to the Object:

$$\sum \vec{F} = \vec{0}\,\mathrm{N} \tag{1.83}$$

In this specific situation the sum of forces is

$$\vec{F}_{\mathrm{G}} + \vec{f} + \vec{n} + \vec{P} = \vec{0}\,\mathrm{N} \tag{1.84}$$

The components of this constraint are the equations we must solve. The x-component of this equation is:

$$F_{G,x} + f_x + n_x + P_x = 0\,\mathrm{N} \tag{1.85}$$
$$(0\,\mathrm{N}) + (+f) + (0\,\mathrm{N}) + (-108.3\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.86}$$

The y-component of this equation is:

$$F_{G,y} + f_y + n_y + P_y = 0\,\mathrm{N} \tag{1.87}$$
$$(-882.9\,\mathrm{N}) + (0\,\mathrm{N}) + (+n) + (+62.5\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.88}$$

Step 5: Solving the x-component equation for the magnitude of the friction:

$$(0\,\mathrm{N}) + (+f) + (0\,\mathrm{N}) + (-108.3\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.89}$$
$$f = +108.3\,\mathrm{N} \tag{1.90}$$

Note that, if we had assumed the incorrect direction for the friction, we would have obtained a negative number at this step. Since a magnitude must be positive, that would have signaled that something need to be corrected in some previous step.

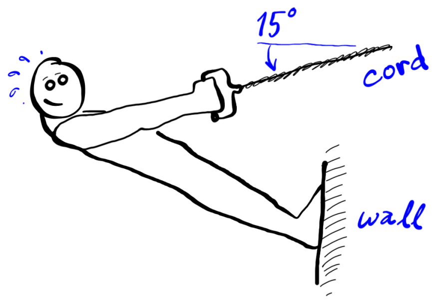
Solving the y-component equation for the magnitude of the normal:

$$(-882.9\,\mathrm{N}) + (0\,\mathrm{N}) + (+n) + (+62.5\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.91}$$
$$n = +820.4\,\mathrm{N} \tag{1.92}$$

As expected from the sum of forces check done with the Free-Body Diagram, the normal is less than the weight since (the reaction to) the push acts upwards on the Object (me).

Answer: The normal acting on me is 820.4 N upwards, and the frictional force acting on me is 108.3 N towards the right.

Problem 1.2.11: An acrobat of weight 700 N is practicing a maneuver, suspending themselves from a vertical wall, as shown in the picture. The rope suspending them makes an angle of 15° with the horizontal, and has a tension of 1160 N. What is the magnitude and direction of the friction exerted on the acrobat's feet by the wall.



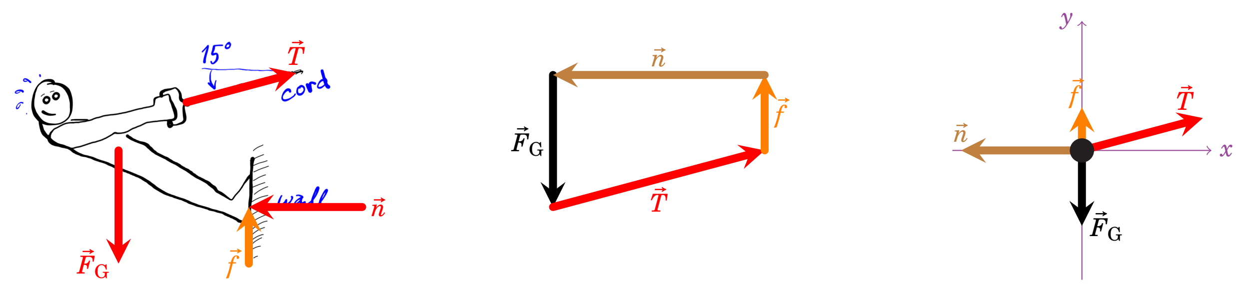
Solution:

Step 0: The object is the acrobat.

Step 1: The acrobat is interacting with the Earth, the rope, and the wall. The forces acting on them are

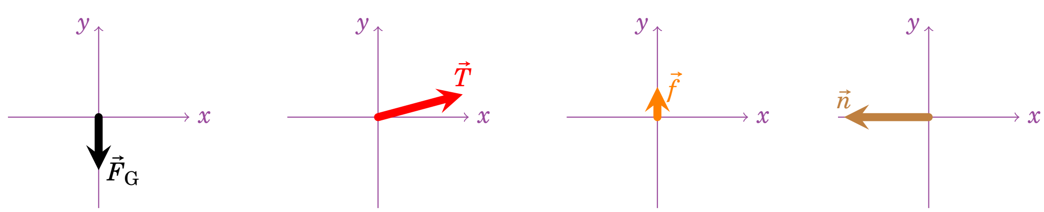
- gravity,
- the contact force with the wall (the normal) pointing horizontally towards the left,
- friction between their feet and the wall,
- the tension of the rope pulling to the right and slightly upwards with magnitude 1160 N.

Step 2: The Free-Body Diagram and check of sum of forces:



Gravity, we know, acts straight vertically down. The tension in the rope pulls to the right and slightly upwards. The normal force acts perpendicular to the surface of contact (the wall) and points in the direction opposite the object would move if the surface were not there. The rope is pulling the acrobat towards the right, so the normal points towards the left. If there were no friction we could imagine the acrobat's feet would slide towards the floor, so we guess that friction points upwards along the wall (the direction is something we will solve for).

Step 3: Find the components of each of the four forces. Note that we do not know which way friction points (upwards or downwards) but we do know that it is parallel to the wall.



$$F_{G,x} = 0\,\text{N} \qquad F_{G,y} = -700\,\text{N}$$

$$T_x = (1160\,\text{N})\cos 15° = +1121\,\text{N} \qquad T_y = (1160\,\text{N})\sin 15° = +300\,\text{N}$$

$$f_x = 0\,\text{N} \qquad f_y = ?$$

$$n_x = -n \qquad n_y = 0\,\text{N}$$

Step 4: Newton's First Law for this problem is

$$\vec{F}_G + \vec{T} + \vec{f} + \vec{n} = \vec{0}\,\text{N} \tag{1.93}$$

The x-component (horizontal) of this equation is

$$F_{G,x} + T_x + f_x + n_x = 0\,\mathrm{N} \tag{1.94}$$

$$(0\,\mathrm{N}) + (+1121\,\mathrm{N}) + (0\,\mathrm{N}) + (-n) = 0\,\mathrm{N} \tag{1.95}$$

The y-component (vertical) of this equation is

$$F_{G,y} + T_y + f_y + n_y = 0\,\mathrm{N} \tag{1.96}$$

$$(-700\,\mathrm{N}) + (+300\,\mathrm{N}) + (f_y) + (0\,\mathrm{N}) = 0\,\mathrm{N} \tag{1.97}$$

Step 5: Solve.

Solving the x-component equation for the magnitude of the normal we find that $n = 1121\,\mathrm{N}$. (Why is the normal not equal to the acrobat's weight? Because it is the tension in the rope – not gravity – that is pulling the acrobat against the wall.)

Solving the y-component equation for the vertical component (parallel to the wall) of the friction we find $f_y = +400\,\mathrm{N}$. That this is positive means that friction points upwards. The magnitude of friction is thus $400\,\mathrm{N}$.

Answer: The frictional force acting on the acrobat is is $400\,\mathrm{N}$ vertically upwards.

1.3 Pulleys

A pulley is a machine that changes the *direction* of a rope but not the tension. The following exercises investigate the mechanical consequences of that fact. The context, as always, is static equilibrium.

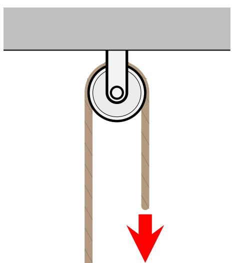
In each of the exercises below find the tension in each rope, when possible. Where asked, find the unknown externally applied force (appearing as a red vector in the diagrams).

To solve for the tension trace along the rope and treat each pulley as if it was its own object. Each pulley being in static equilibrium means that the sum of forces acting on each pulley must sum to zero, individually.

Pay very close attention to how the system of pulleys attaches to weights in the problem; you will quite often find that the tension in the rope is only a fraction of the object's weight.

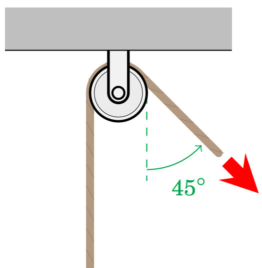
1.3.1 A Single Pulley

Exr 1.3.01



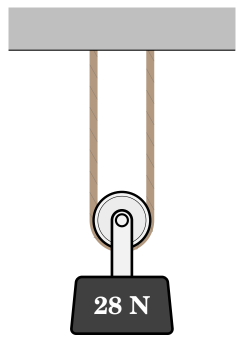
$T = 12\,\text{N} = F$.

Exr 1.3.02



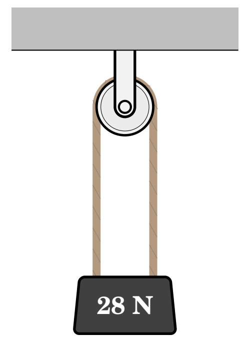
$T = 12\,\text{N} = F$.

Exr 1.3.03



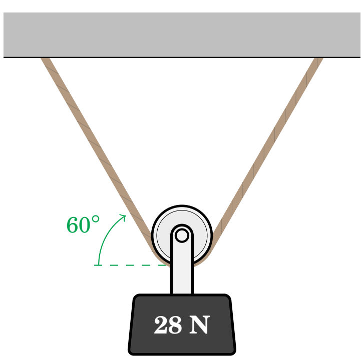
$T = 14\,\text{N}$.

Exr 1.3.04



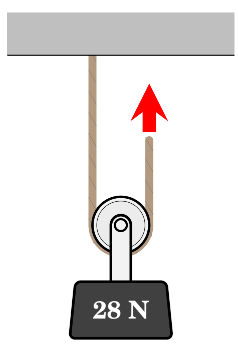
$T = 14\,\text{N}$.

Exr 1.3.05



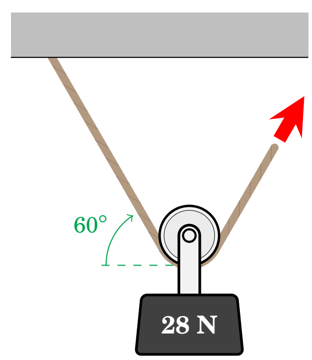
$T = 16.17\,\text{N}$.

Exr 1.3.06



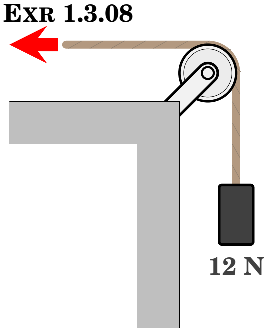
$T = 14\,\text{N} = F$.

Exr 1.3.07



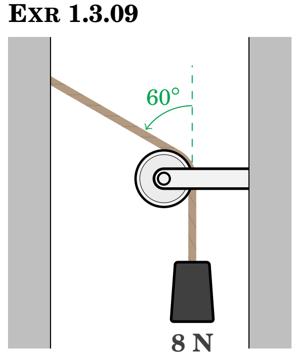
$T = 16.17\,\text{N} = F$.

Exr 1.3.08



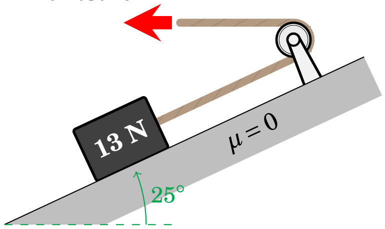
$T = 12\,\text{N} = F$.

Exr 1.3.09



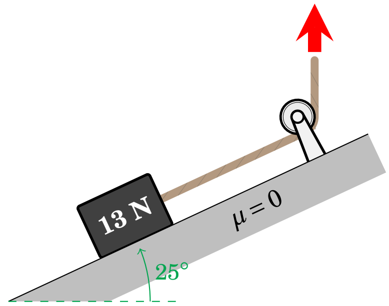
$T = 8\,\text{N}$.

Exr 1.3.10



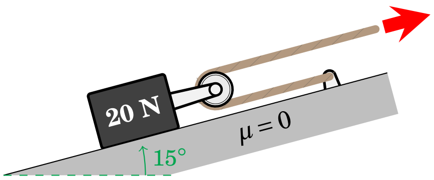
$T = 5.49\,\mathrm{N} = F$.

Exr 1.3.11



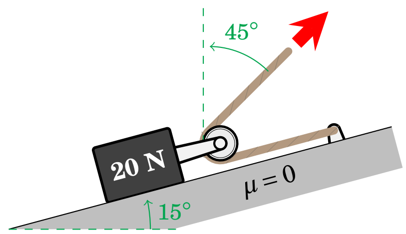
$T = 5.49\,\mathrm{N} = F$.

Exr 1.3.12



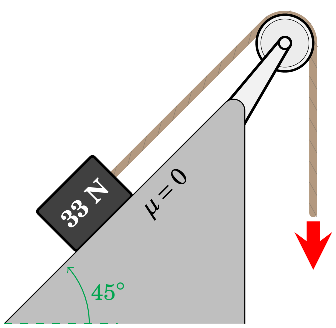
$T = 2.59\,\mathrm{N} = F$.

Exr 1.3.13



$T = 2.77\,\mathrm{N} = F$.

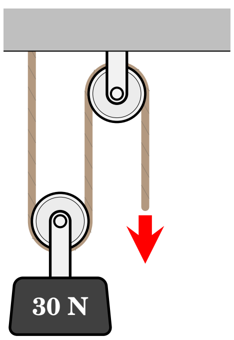
Exr 1.3.14



$T = 23.33\,\mathrm{N} = F$.

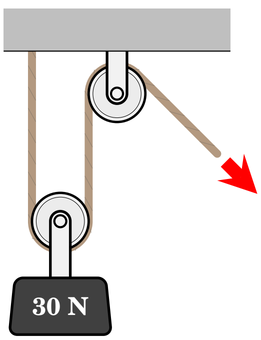
1.3.2 Two Pulleys

Exr 1.3.15



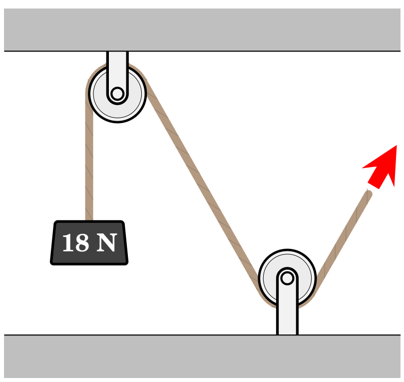
$T = 15\,\mathrm{N} = F$.

Exr 1.3.16



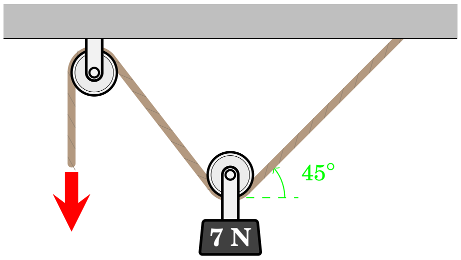
$T = 15\,\mathrm{N} = F$.

Exr 1.3.17



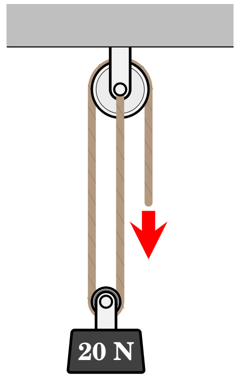
$T = 18\,\mathrm{N} = F$.

Exr 1.3.18



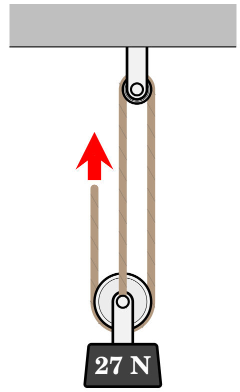
$T = 4.94\,\mathrm{N} = F$.

Exr 1.3.19



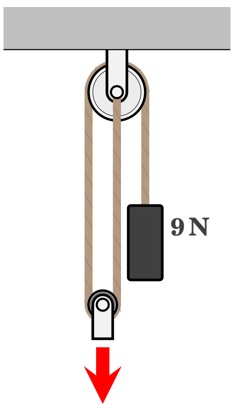
$T = 10\,\mathrm{N} = F$.

Exr 1.3.20



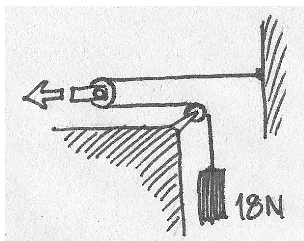
$T = 9\,\mathrm{N} = F$.

Exr 1.3.21



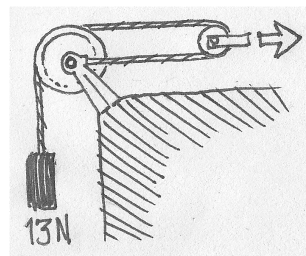
$T = 9\,\mathrm{N}$,
$F = 18\,\mathrm{N}$.

Exr 1.3.22



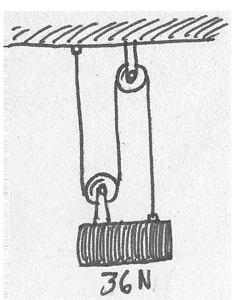
$T = 18\,\mathrm{N}$,
$F = 36\,\mathrm{N}$.

Exr 1.3.23



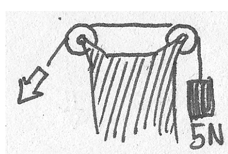
$T = 13\,\mathrm{N}$,
$F = 26\,\mathrm{N}$.

Exr 1.3.24



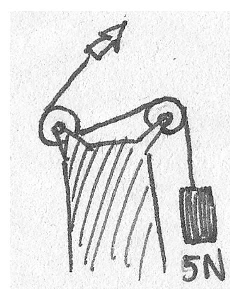
$T = 12\,\mathrm{N}$.

Exr 1.3.25



$T = 5\,\mathrm{N}$.

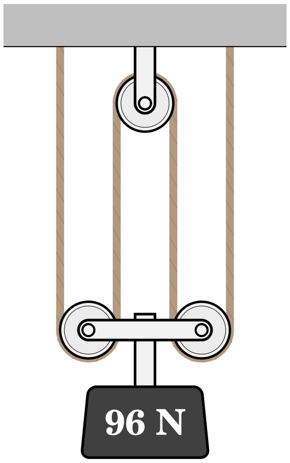
Exr 1.3.26



$T = 5\,\mathrm{N}$.

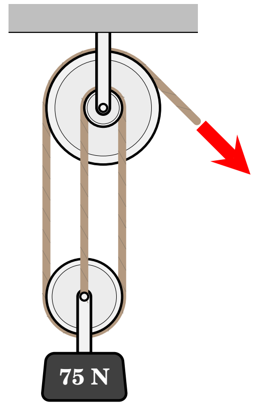
1.3.3 Three or more Pulleys

EXR 1.3.27



$T = 24\,\text{N}$.

EXR 1.3.28

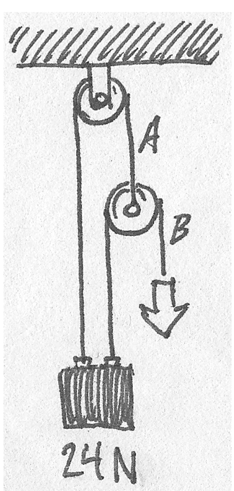


$T = 25\,\text{N} = F$.

1.3.4 Multiple Ropes

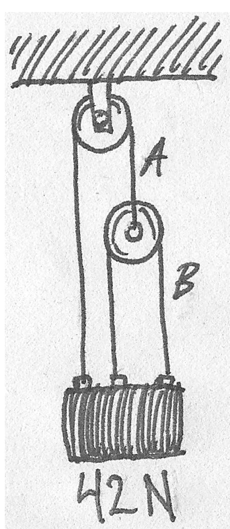
In each of these systems find the tension in each of the ropes.

EXR 1.3.29



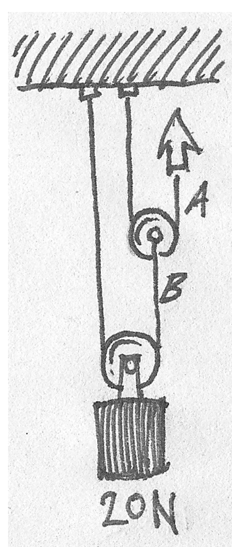
$T_A = 16\,\text{N}$,
$T_B = 8\,\text{N} = F$.

EXR 1.3.30



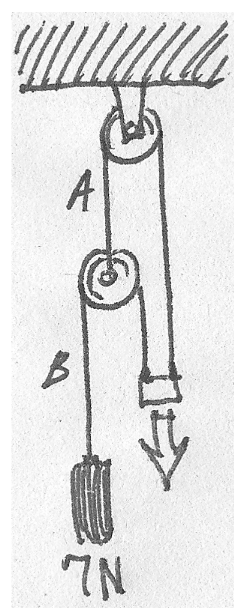
$T_A = 21\,\text{N}$,
$T_B = 10.5\,\text{N}$.

EXR 1.3.31



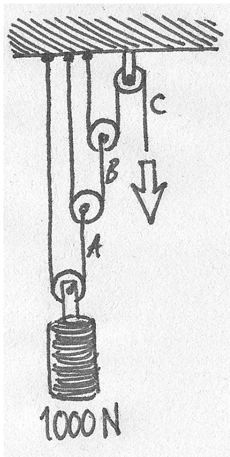
$T_A = 5\,\text{N} = F$,
$T_B = 10\,\text{N}$.

EXR 1.3.32



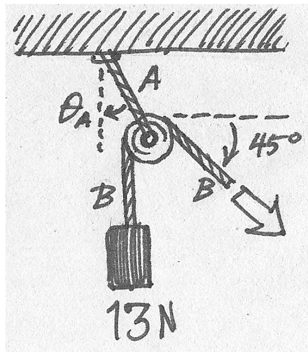
$T_A = 14\,\text{N}$,
$T_B = 7\,\text{N}$,
$F = 21\,\text{N}$.

EXR 1.3.33



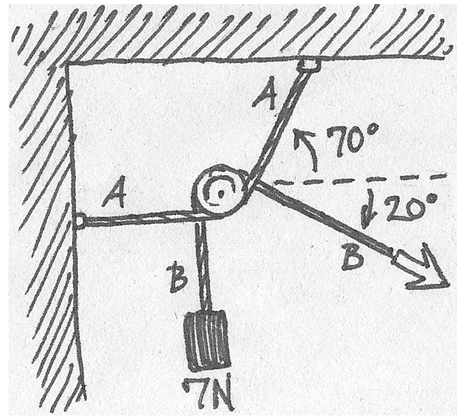
$T_A = 500\,\mathrm{N}$,
$T_B = 250\,\mathrm{N}$,
$T_C = 125\,\mathrm{N} = F$.

EXR 1.3.34



$T_B = 13\,\mathrm{N} = F$,
$\theta_A = 22.5°$,
$T_A = 24.02\,\mathrm{N}$.

EXR 1.3.35



$T_B = 7\,\mathrm{N} = F$,
$T_A = 10.00\,\mathrm{N}$.

Chapter 2

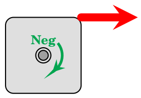
Torques

2.1 Torque : Qualitative Exercises

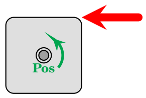
2.1.1 Sign of Torque

In each of the following exercises find the *sign* of the torque about the pivot produced by the single applied force. If the applied force produces a torque of magnitude zero, say so. Remember that the sign of the torque is the sign of the rotation that would happen *if* that torque was the only one being applied. (The sign of rotation follows the sign convention of angles, with counter-clockwise being positive and clockwise being negative.)

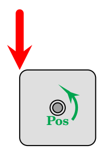
Exr 2.1.01



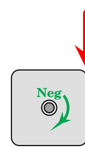
Exr 2.1.02



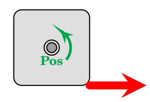
Exr 2.1.03



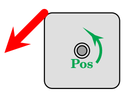
Exr 2.1.04



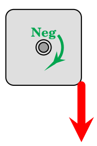
Exr 2.1.05



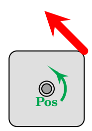
Exr 2.1.06



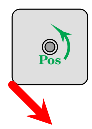
Exr 2.1.07



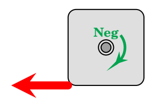
Exr 2.1.08



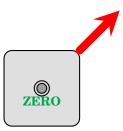
Exr 2.1.09



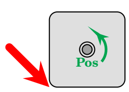
Exr 2.1.10



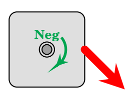
Exr 2.1.11



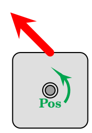
Exr 2.1.12



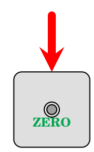
Exr 2.1.13



Exr 2.1.14



Exr 2.1.15



2.1.2 Line of Action

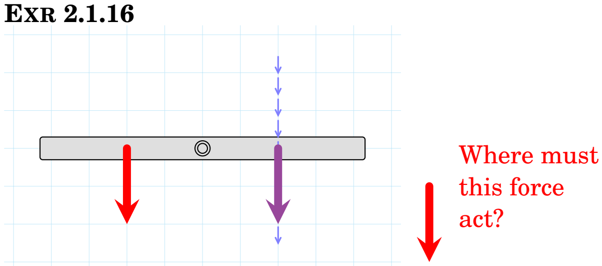
In the following exercises we use the *line of action* to reason qualitatively about the magnitude of torque. The context is static equilibrium, in which $\sum \vec{\tau} = \vec{0}\,\mathrm{N{\cdot}m}$ about any axis. With the object in the xy-plane (the page), the z-axis will be through the chosen pivot (perpendicular to the page), and $\sum \tau_z = 0\,\mathrm{N{\cdot}m}$ about that axis.

In the cases where we are given a force, the exercise is to find the line of action along which it must act so that the object remains in static equilibrium. In the cases where we are given a line of action, the exercise is to find (qualitatively) the force that would, acting along that line, keep the object in static equilibrium.

Thin rectangular rod

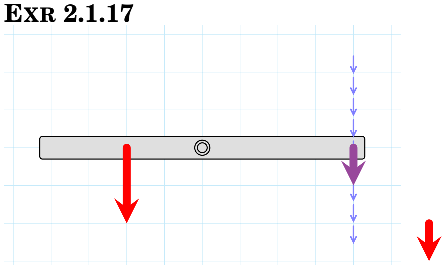
In this first set of exercises the pivot will be located at the center of gravity of the object. Neither the force of gravity nor the contact force of the pivot contribute a torque, so their magnitude and direction are unimportant. The forces do still exist but, to simplify the diagrams, neither will be shown.

Exr 2.1.16

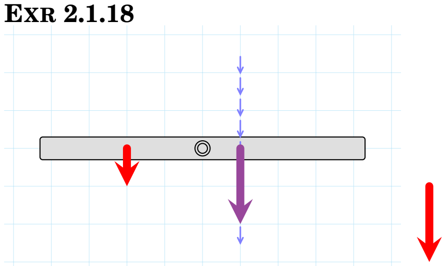


For the torques to cancel they must be of equal magnitude but opposite sign. The force to be applied is equal in magnitude to the existing force. Thus the distance d must be the same for both forces. Both forces point in the same direction, so they must be placed on opposite sides of the pivot for the signs of their torques to be opposite.

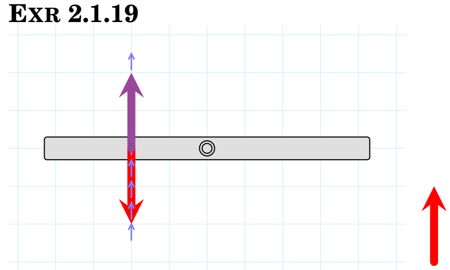
Exr 2.1.17



Exr 2.1.18

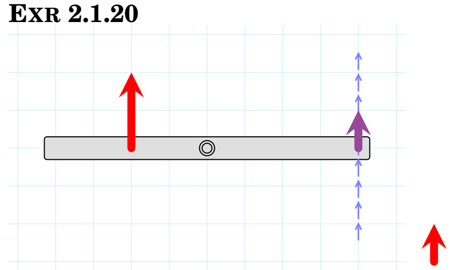


Exr 2.1.19

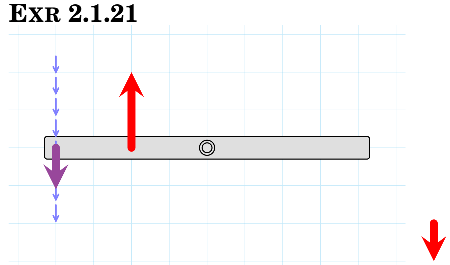


Since the forces point in opposite directions, so they must be placed on the same side of the pivot for the signs of their torques to be opposite.

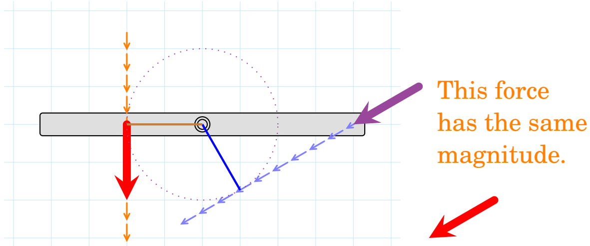
Exr 2.1.20



Exr 2.1.21

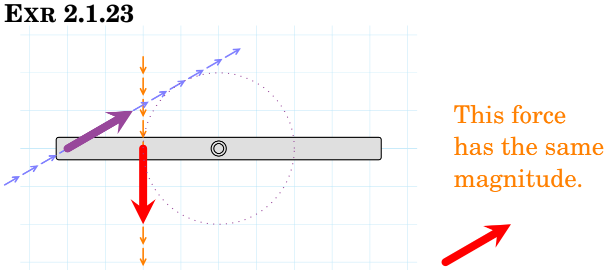


Exr 2.1.22

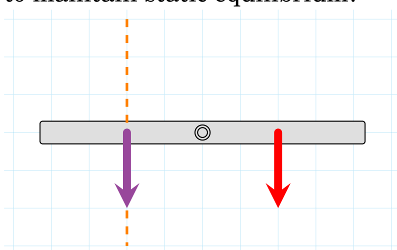


For the torques to cancel they must be of equal magnitude but opposite sign. Since both forces have the same component perpendicular to the object, they must be on opposite sides of the pivot for their torques to be of opposite sign. The force to be applied is equal in magnitude to the existing force. Thus the distance d must be the same for both forces. The line of action of the applied force must pass the same distance from the pivot. (Notice how the lines of action are tangent to a common circle.)

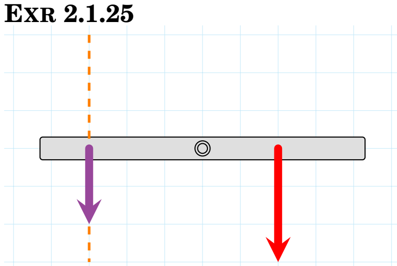
Exr 2.1.23



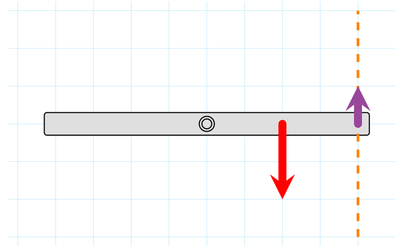
Exr 2.1.24
What force must be placed on the dashed orange line to maintain static equilibrium?



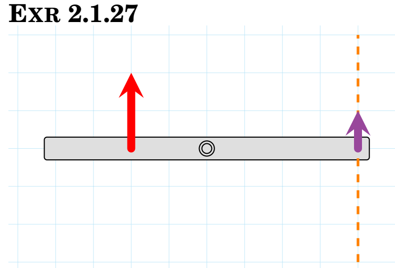
Exr 2.1.25



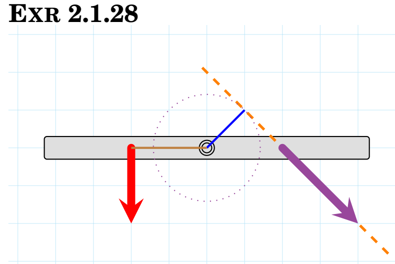
Exr 2.1.26



Exr 2.1.27

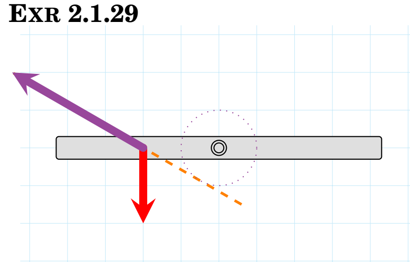


Exr 2.1.28



The line of action for the applied force passes closer to the pivot, so it must have a larger magnitude.

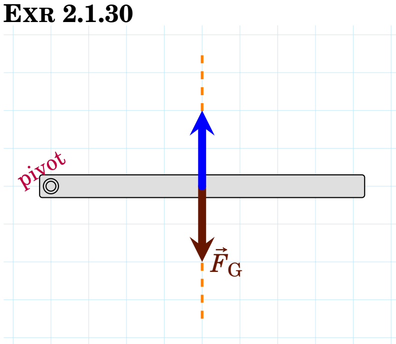
Exr 2.1.29



The line of action for the applied force is half the distance from the pivot. The the applied force must be twice as large for the torques to cancel.

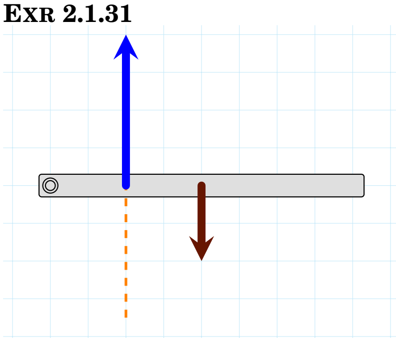
In this next set of exercises the pivot is ***not*** at the center of gravity of the object. This means that gravity will exert a torque about the pivot on the object. (The force of gravity $\vec{F}_G$ is coloured dark brown in the diagrams.) The goal will be to find or place the force that will maintain equilibrium.

Exr 2.1.30

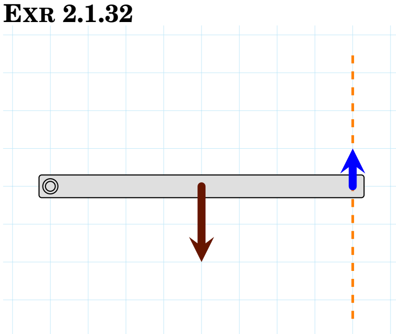


The applied force and gravity are on the same side of the pivot. For their torques to be of opposite signs they must point in opposite directions.

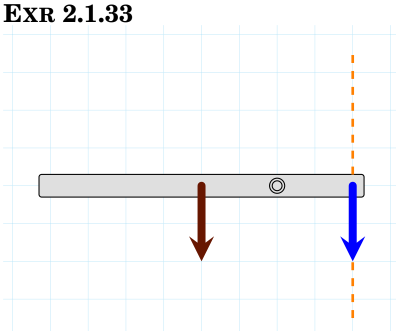
Exr 2.1.31



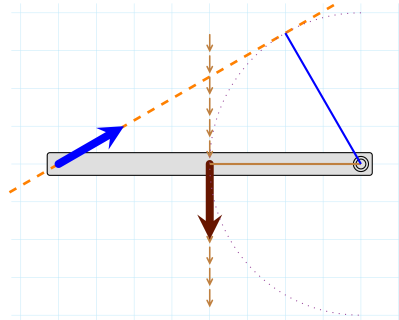
Exr 2.1.32



Exr 2.1.33

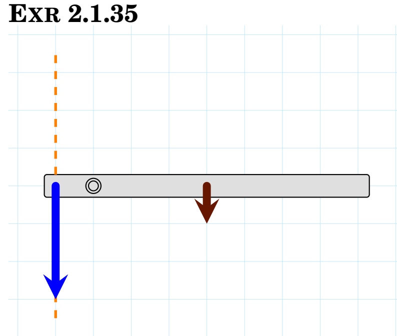


Exr 2.1.34



The line of action for each force is equidistant from the pivot. Thus the applied force must have the same magnitude for the torques to cancel.

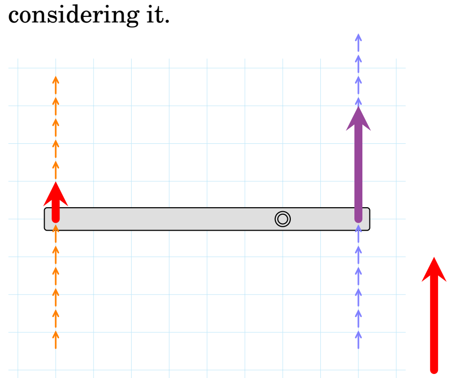
Exr 2.1.35



The applied force is closer to the pivot, so it must be proportionally larger for the torques to cancel.

Exr 2.1.36

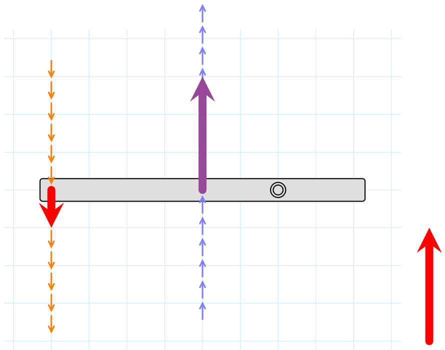
In this exercise gravity is present, but is so much smaller than the other forces present that we omit considering it.



The magnitude of the applied force is much larger than the existing force, so it must act closer to the pivot. Since both forces point in the same direction, they must be on opposite sides of the pivot for the torques to cancel.

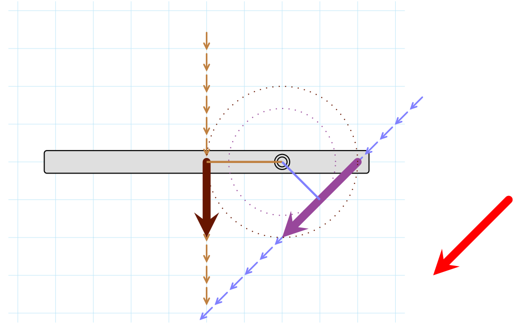
Exr 2.1.37

In this exercise gravity is present, but is so much smaller than the other forces present that we omit considering it.



The magnitude of the applied force is much larger than the existing force, so it must act closer to the pivot. Since both forces point in opposite directions, they must be on the same side of the pivot for the torques to cancel.

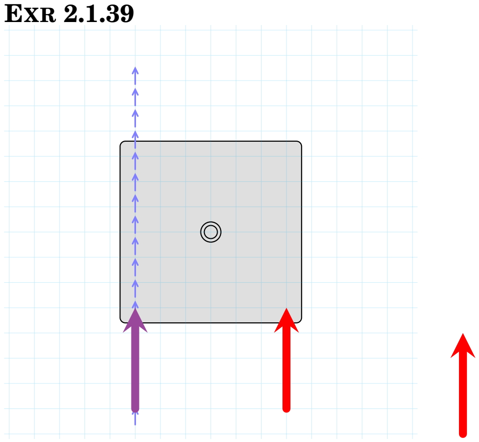
Exr 2.1.38



For the torques to cancel they must be of equal magnitude but opposite sign. Since both forces have the same component perpendicular to the object, they must be on opposite sides of the pivot for their torques to be of opposite sign. The magnitude of the applied force is larger than the object's weight. Thus the line of action of the applied force must pass closer to the pivot for the torques to cancel.

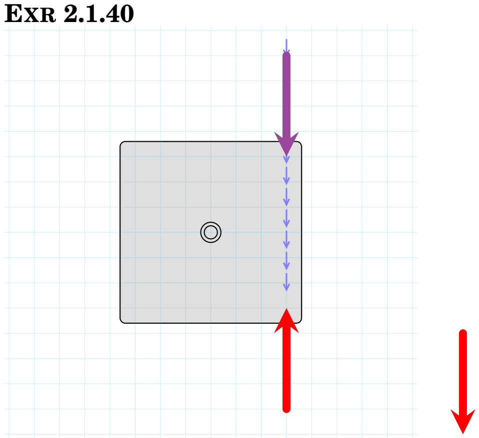
Square and rectangular objects

Exr 2.1.39



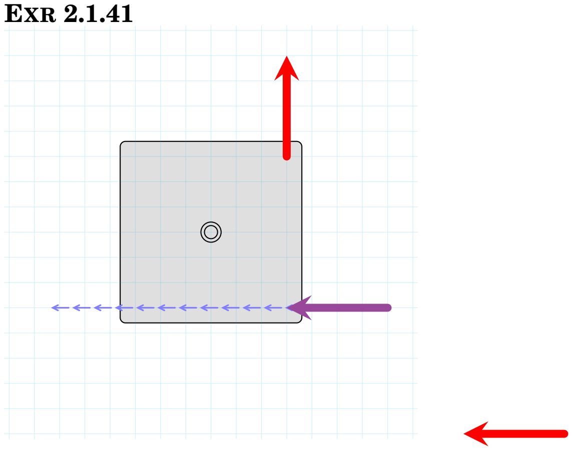
The applied force can be anywhere on the line of action along the left edge of the object and produce the same torque.

Exr 2.1.40



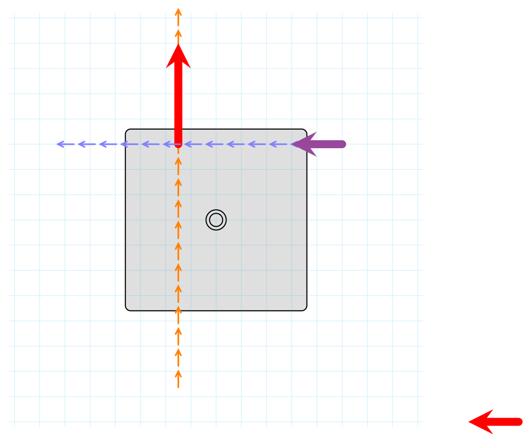
The applied force can be anywhere on the line of action along the right edge of the object and produce the same torque. (Note how both forces are on the same line of action, but in opposite directions.)

Exr 2.1.41



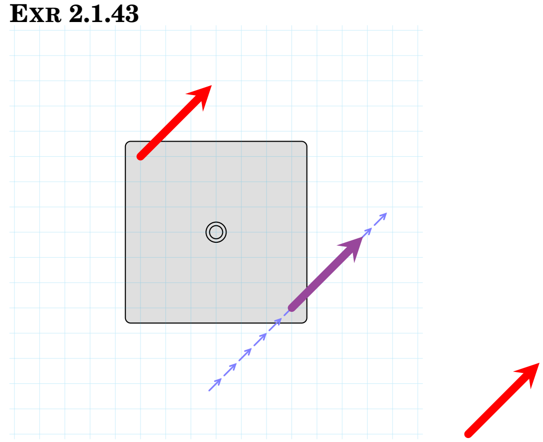
The applied force can be anywhere on the line of action along the bottom edge of the object and produce the same torque.

Exr 2.1.42

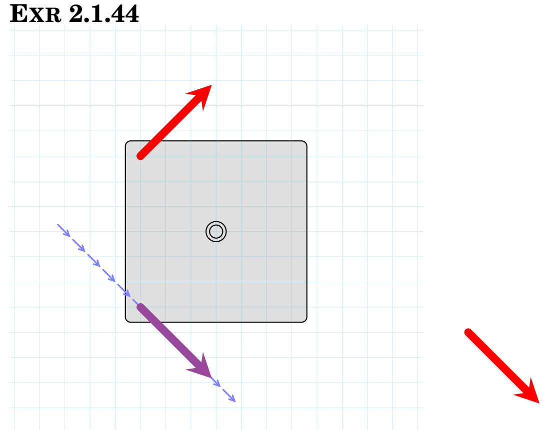


Since the applied force is small than the existing force its line of action must pass closer to the pivot. The applied force can be anywhere on the line of action along the top edge of the object and produce the same torque.

Exr 2.1.43

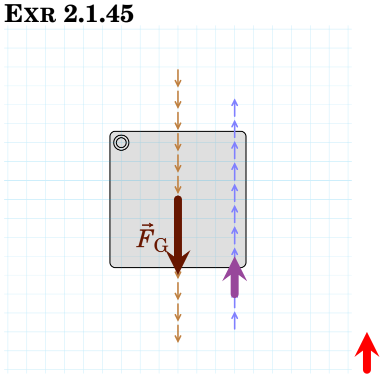


Exr 2.1.44

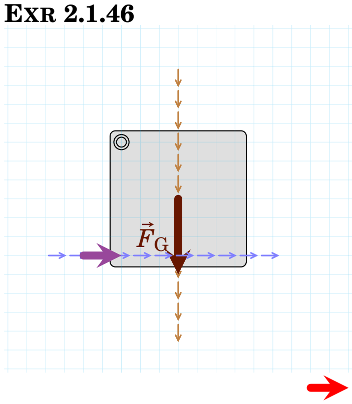


In this next set of exercises the pivot is ***not*** at the center of gravity of the object. This means that gravity will exert a torque about the pivot on the object. (The force of gravity $\vec{F}_G$ is coloured dark brown in the diagrams.) The goal will be to find or place the force that will maintain equilibrium.

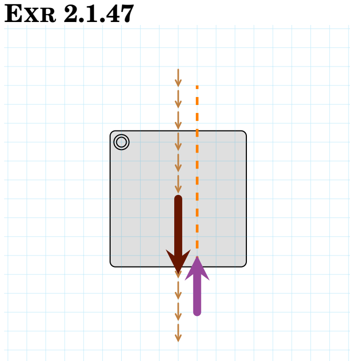
Exr 2.1.45



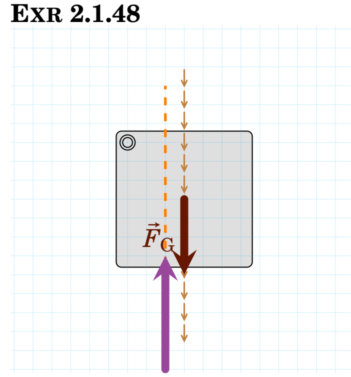
Exr 2.1.46



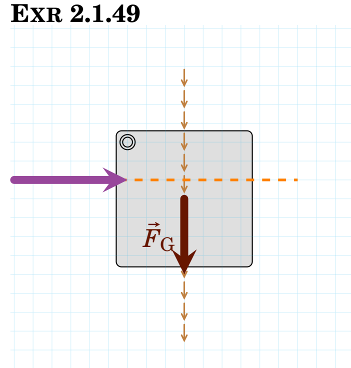
Exr 2.1.47



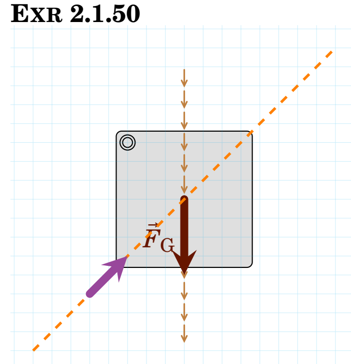
Exr 2.1.48



Exr 2.1.49

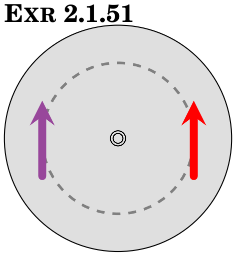


Exr 2.1.50

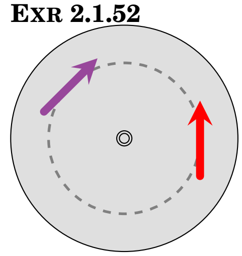


Circular objects

Exr 2.1.51



Exr 2.1.52



Irregularly-shaped objects

▲**FIX:** TO BE WRITTEN

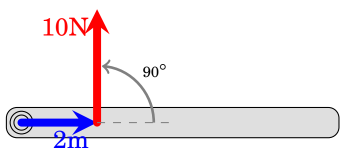
2.2 Torque : Quantitative Exercises

2.2.1 Single applied Force

In each of these exercises determine:
(1) the angle θ from the direction of $\vec{r}$ to the direction of $\vec{F}$, and hence the sign of τ_z;
(2) the component $F_\perp$ of the force that exerts a torque; and
(3) the z-component of the torque (τ_z) exerted about the pivot by the applied force.

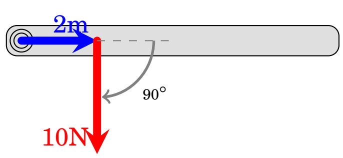
Right Angles

Exr 2.2.01



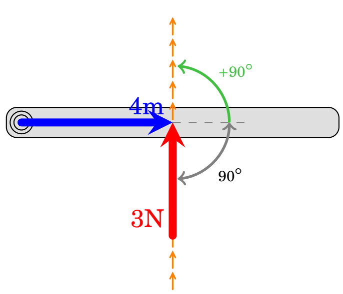
(1) The angle is given as $\theta = +90°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 10\,\mathrm{N} \times \sin(+90°) = +10\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+10\,\mathrm{N})(2\,\mathrm{m}) = +20\,\mathrm{N{\cdot}m}$.

Exr 2.2.02



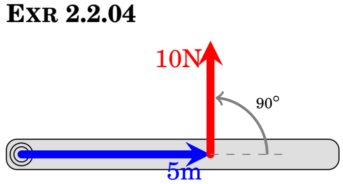
(1) The angle is given as $\theta = -90°$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F \sin\theta = 10\,\mathrm{N} \times \sin(-90°) = -10\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-10\,\mathrm{N})(2\,\mathrm{m}) = -20\,\mathrm{N{\cdot}m}$.

Exr 2.2.03



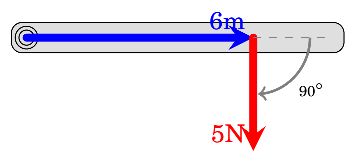
(1) The angle shown is 90°, and looks like it is clockwise. But, measured from the direction of $\vec{r}$ towards the direction of the force's line of action, the angle is $\theta = +90°$ (counter-clockwise). So the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 3\,\mathrm{N} \times \sin(+90°) = +3\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+3\,\mathrm{N})(4\,\mathrm{m}) = +12\,\mathrm{N{\cdot}m}$.

Exr 2.2.04



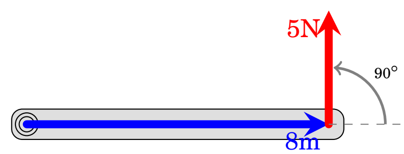
(1) The angle is given as $\theta = +90°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 10\,\mathrm{N} \times \sin(+90°) = +10\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+10\,\mathrm{N})(5\,\mathrm{m}) = +50\,\mathrm{N{\cdot}m}$.

Exr 2.2.05



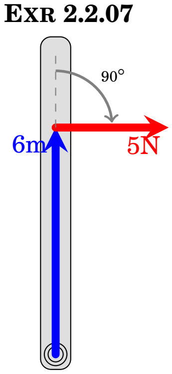
(1) The angle is given as $\theta = -90°$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F \sin\theta = 5\,\mathrm{N} \times \sin(-90°) = -5\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-5\,\mathrm{N})(6\,\mathrm{m}) = -30\,\mathrm{N{\cdot}m}$.

Exr 2.2.06



(1) The angle is given as $\theta = +90°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 5\,\mathrm{N} \times \sin(+90°) = +5\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+5\,\mathrm{N})(8\,\mathrm{m}) = +40\,\mathrm{N{\cdot}m}$.

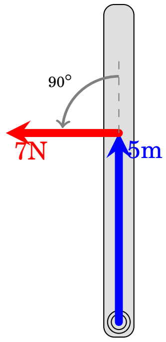
Exr 2.2.07



(1) The angle is given as $\theta = -90°$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F \sin\theta = 5\,\mathrm{N} \times \sin(-90°) = -5\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-5\,\mathrm{N})(6\,\mathrm{m}) = -30\,\mathrm{N{\cdot}m}$.

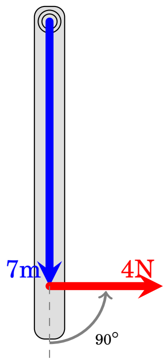
Notice how, in this exercise, we have $r_y > 0$ m and $F_x > 0\,\mathrm{N}$, but $\tau_z < 0\,\mathrm{N{\cdot}m}$ due to the relation between their directions. Remember this example for how it shows that the vectors (specifically the signs of their components) *independent of each other* do not give the sign of the torque. The sign of τ_z is determined by how the directions of $\vec{r}$ and $\vec{F}$ *relate* to each other.

EXR 2.2.08



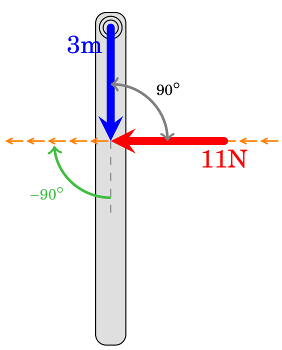
(1) The angle is given as $\theta = +90^\circ$,
so the torque τ_z will be positive.
(2) $F_\perp = F\,\sin\theta = 7\,\mathrm{N} \times \sin(+90^\circ) = +7\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+7\,\mathrm{N})(5\,\mathrm{m}) = +35\,\mathrm{N{\cdot}m}$.

EXR 2.2.09



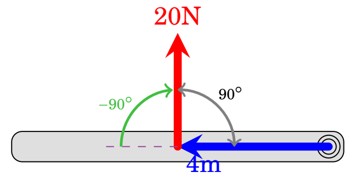
(1) The angle is given as $\theta = +90^\circ$,
so the torque τ_z will be positive.
(2) $F_\perp = F\,\sin\theta = 4\,\mathrm{N} \times \sin(+90^\circ) = +4\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+4\,\mathrm{N})(7\,\mathrm{m}) = +28\,\mathrm{N{\cdot}m}$.

EXR 2.2.10



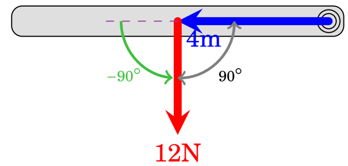
(1) Measured from the direction of $\vec{r}$ towards the direction of the line of action, the angle is $\theta = -90^\circ$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F\,\sin\theta = 11\,\mathrm{N} \times \sin(-90^\circ) = -11\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-11\,\mathrm{N})(3\,\mathrm{m}) = -33\,\mathrm{N{\cdot}m}$.

EXR 2.2.11



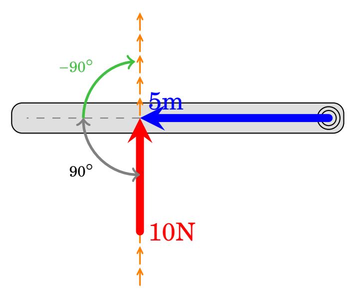
(1) Measured from the direction of $\vec{r}$ towards the direction of the line of action, the angle is $\theta = -90^\circ$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F\,\sin\theta = 20\,\mathrm{N} \times \sin(-90^\circ) = -20\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-20\,\mathrm{N})(4\,\mathrm{m}) = -80\,\mathrm{N{\cdot}m}$.

EXR 2.2.12



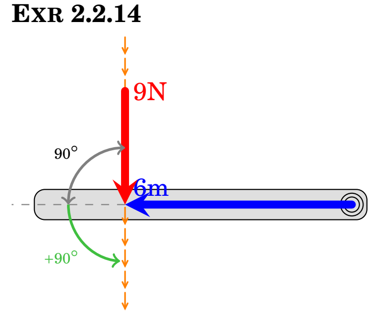
(1) Measured from the direction of $\vec{r}$ towards the direction of the line of action, the angle is $\theta = +90^\circ$ (counter-clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F\,\sin\theta = 12\,\mathrm{N} \times \sin(+90^\circ) = +12\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+12\,\mathrm{N})(4\,\mathrm{m}) = +48\,\mathrm{N{\cdot}m}$.

EXR 2.2.13



(1) Measured from the direction of $\vec{r}$ towards the direction of the line of action, the angle is $\theta = -90^\circ$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F\,\sin\theta = 10\,\mathrm{N} \times \sin(-90^\circ) = -10\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-10\,\mathrm{N})(5\,\mathrm{m}) = -50\,\mathrm{N{\cdot}m}$.

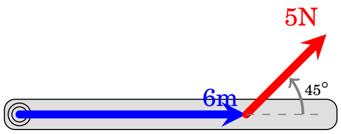
EXR 2.2.14



(1) Measured from the direction of $\vec{r}$ towards the direction of the line of action, the angle is $\theta = +90^\circ$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F\,\sin\theta = 9\,\mathrm{N} \times \sin(+90^\circ) = +9\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+9\,\mathrm{N})(6\,\mathrm{m}) = +54\,\mathrm{N{\cdot}m}$.

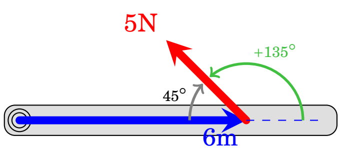
Non-Right Angles

Exr 2.2.15



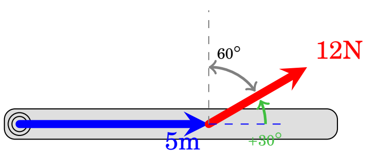
(1) The angle is given as $\theta = +45°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 5\,\mathrm{N} \times \sin(+45°) = +3.536\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+3.536\,\mathrm{N})(6\,\mathrm{m}) = +21.2\,\mathrm{N{\cdot}m}$.

Exr 2.2.16



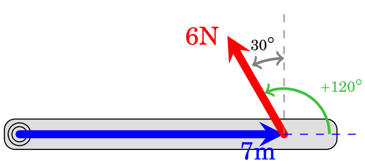
(1) In the diagram the angle is found to be $\theta = +135°$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 5\,\mathrm{N} \times \sin(+135°) = +3.536\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+3.536\,\mathrm{N})(6\,\mathrm{m}) = +21.2\,\mathrm{N{\cdot}m}$.

Exr 2.2.17



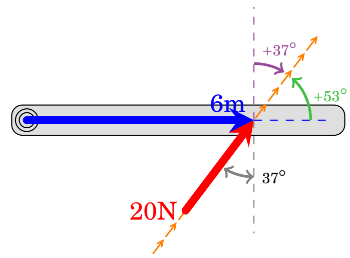
From the diagram we find that the angle is $\theta = +30°$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 12\,\mathrm{N} \times \sin(+30°) = +6\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+6\,\mathrm{N})(5\,\mathrm{m}) = +30\,\mathrm{N{\cdot}m}$.

Exr 2.2.18



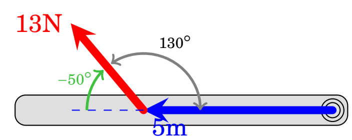
(1) From the diagram we find that the angle is $\theta = +120°$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 6\,\mathrm{N} \times \sin(+120°) = +5.196\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+5.196\,\mathrm{N})(7\,\mathrm{m}) = +36.4\,\mathrm{N{\cdot}m}$.

Exr 2.2.19



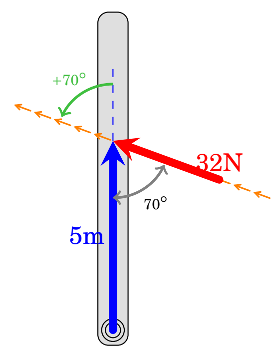
(1) From the diagram we find that the angle is $\theta = +53°$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 20\,\mathrm{N} \times \sin(+53°) = +15.97\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+15.97\,\mathrm{N})(6\,\mathrm{m}) = +96\,\mathrm{N{\cdot}m}$.

Exr 2.2.20



(1) From the diagram we find that the angle is $\theta = -50°$ (clockwise), so the torque τ_z will be negative.
(2) $F_\perp = F \sin\theta = 13\,\mathrm{N} \times \sin(-50°) = -9.959\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (-9.959\,\mathrm{N})(5\,\mathrm{m}) = -50\,\mathrm{N{\cdot}m}$.

Exr 2.2.21

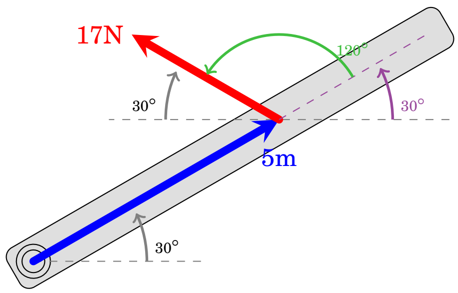


(1) From the diagram we find that the angle is $\theta = +70°$ (counter-clockwise), so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 32\,\mathrm{N} \times \sin(+70°) = +30.07\,\mathrm{N}$.
(3) $\tau_z = F_\perp r = (+30.07\,\mathrm{N})(5\,\mathrm{m}) = +150\,\mathrm{N{\cdot}m}$.

General Cases

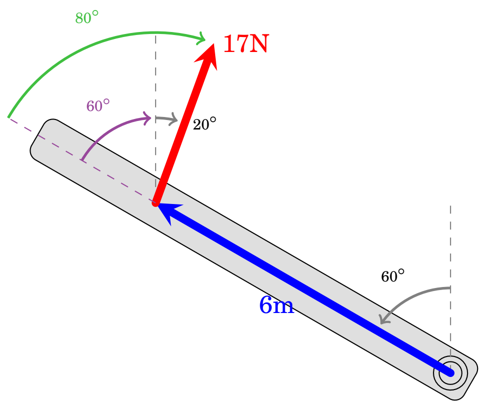
In these cases you must solve the geometry to find the angle θ from the direction of $\vec{r}$ towards the direction of $\vec{F}$. Recall how to identify complementary angles, and rules like the transverse-parallel theorem. Be very careful to get the sign of θ correct. These exercises may require a few steps to solve the geometry. Be sure to draw larger diagrams than usual so that you can label the parts of the geometry clearly.

Exercise 2.2.22



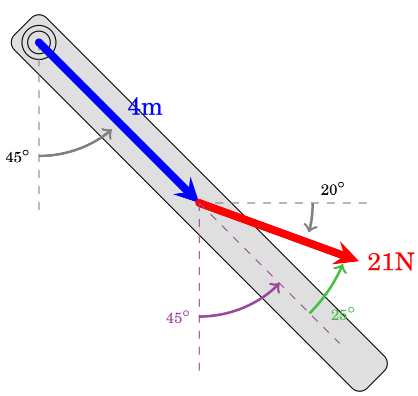
(1) In the diagram the angle is found to be $\theta = +120°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 17\,\mathrm{N} \times \sin(+120°) = +14.722\,\mathrm{N}$
(3) $\tau_z = F_\perp r = (+14.722\,\mathrm{N})(5\,\mathrm{m}) = +73.6\,\mathrm{N{\cdot}m}$.

Exercise 2.2.23



(1) In the diagram the angle is found to be $\theta = -80°$, so the torque τ_z will be negative.
(2) $F_\perp = F \sin\theta = 17\,\mathrm{N} \times \sin(-80°) = -16.742\,\mathrm{N}$
(3) $\tau_z = F_\perp r = (-16.742\,\mathrm{N})(6\,\mathrm{m}) = -100.\,\mathrm{N{\cdot}m}$.

Exercise 2.2.24 (Challenge)



(1) In the diagram the angle is found to be $\theta = +25°$, so the torque τ_z will be positive.
(2) $F_\perp = F \sin\theta = 21\,\mathrm{N} \times \sin(+25°) = +8.875\,\mathrm{N}$
(3) $\tau_z = F_\perp r = (+8.875\,\mathrm{N})(4\,\mathrm{m}) = +35.5\,\mathrm{N{\cdot}m}$.

2.3 Solving Problems using *the Process*

The Process

0. Identify the Object!

1. Identify the forces acting on the Object.

2. Draw the Free-Body Diagram, clearly identifying the pivot.

3. Separately, for each force acting on the Object:

- draw the object and the coordinates
- draw the force, placing it on the object where it is acting
- draw the position vector $\vec{r}$ from the pivot to where the force is acting
- determine the components of the force, and the contribution to torque.

4. Use Newton's 1st Law to write the equations to be solved.

5. Solve the equations for the unknown quantities.

2.3.1 Solving for Forces using Torque

A mechanical system is said to be in static equilibrium when the sum of forces is zero and the sum of torques is also zero:

$$\sum F_x = 0\,\mathrm{N} \tag{2.1}$$
$$\sum F_y = 0\,\mathrm{N} \tag{2.2}$$
$$\sum \tau_z = 0\,\mathrm{N{\cdot}m} \tag{2.3}$$

In the exercises below we will begin by practicing how to solve for forces quantitatively using the equilibrium of torques. We will begin with simple geometric shapes, like squares, rectangles and circles. After that we will practice applying the rules of static equilibrium to mechanical objects of various shapes. Follow the steps of *The Process*!

Rectangular Solids

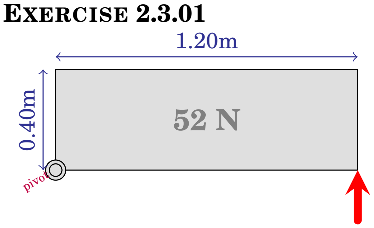
In each of the exercises that follow determine:

(1) Find the magnitude of the applied force that keeps the box in equilibrium; and

(2) Find the magnitude and direction of the force acting at the pivot.

In all cases, unless otherwise indicated, the force of gravity acts at the geometric center of the object. The magnitude of $\vec{F}_{\mathrm{G}}$ is the value printed on the object.

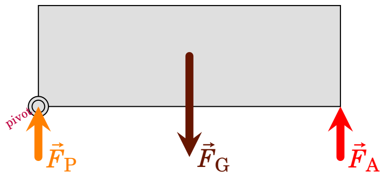
EXERCISE 2.3.01

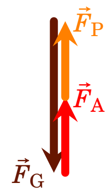


Step 0: The object is the box.

Step 1: There is gravity acting downwards at the center of the object, which exerts a clock-wise torque. And there is the applied upward force acting at the bottom right-hand corner, which applies a counter-clock-wise torque. There may be a force at the pivot, but it does not exert a torque.

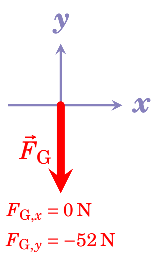
Step 2: The Free-Body Diagram, and the check of the sum of forces is below. Remember that we are only able to guess the force at the pivot on the FBD, and that is after we've tried to make the sum of forces diagram.

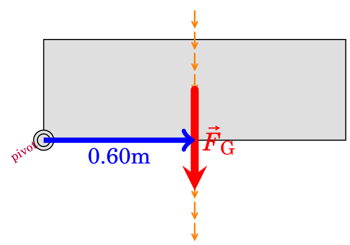




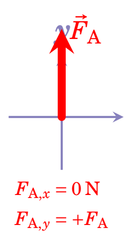
Gravity is vertically downwards and the applied force is vertically upwards. Since the applied force is further from the pivot, to balance the torques we will expect that the applied force is smaller (in magnitude) than gravity. This leads us to conclude that the force at the pivot will have to be vertically upwards, too. (We won't assume this, but will check our final answer against this reasoning.)

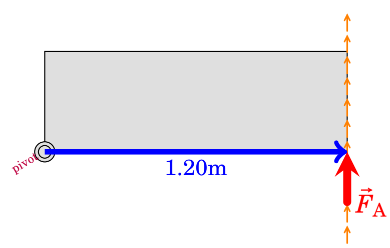
Step 3: The components of the forces, and the contributions to the torque about the pivot are determined below. With the forces being purely vertical, the moment arms are clearly visible on the rectangular object, and we can use "$\tau_z = \pm Fd$" formula to calculate each torque.





$\tau_{{\rm G},z} = -(52\,{\rm N})(0.60{\rm m}) = -31.20\,{\rm N{\cdot}m}$





$\tau_{{\rm A},z} = +F_{\rm A} \times 1.20{\rm m}$

The components of the force at the pivot ($F_{{\rm P},x}$ and $F_{{\rm P},y}$) are both unknowns. This force exerts no torque ($\tau_{{\rm P},z} = 0\,{\rm N{\cdot}m}$) since it is acting *at* the pivot (its moment arm is $d = 0\,{\rm m}$).

Step 4: The conditions for static equilibrium given by Newton's 1st Law are

$$F_{{\rm G},x} + F_{{\rm A},x} + F_{{\rm P},x} = 0\,{\rm N} \qquad 0\,{\rm N} + 0\,{\rm N} + F_{{\rm P},x} = 0\,{\rm N} \tag{2.4}$$

$$F_{{\rm G},y} + F_{{\rm A},y} + F_{{\rm P},y} = 0\,{\rm N} \qquad (-52\,{\rm N}) + (+F_{\rm A}) + F_{{\rm P},y} = 0\,{\rm N} \tag{2.5}$$

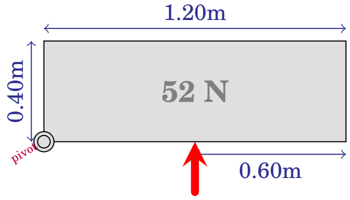
$$\tau_{{\rm G},z} + \tau_{{\rm A},z} + \tau_{{\rm P},z} = 0\,{\rm N{\cdot}m} \qquad (-31.20\,{\rm N{\cdot}m}) + (+F_{\rm A} \times 1.20{\rm m}) + (0\,{\rm N{\cdot}m}) = 0\,{\rm N{\cdot}m} \tag{2.6}$$

Notice how the force at the pivot does not contribute in the torque equation, leaving only one unknown, and making it very easy to solve. This will almost always happen in problems of static equilibrium when we choose to calculate the torques about the pivot.

Step 5: Solving the torque equation we find that $F_{\rm A} = 26.0\,{\rm N}$. Substituting that value in the equation for the sum of the y-components, we find that $F_{{\rm P},y} = +26.0\,{\rm N}$. Evaluating the equation for the sum of the x-components, we see that $F_{{\rm P},x} = 0\,{\rm N}$.

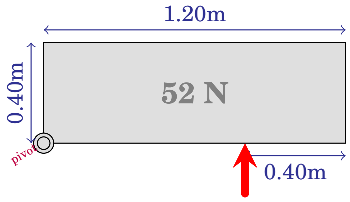
Answer: The applied force that maintains static equilibrium is 26.0 N, vertically upwards. The force at the pivot is (also) 26.0 N, vertically upwards. (Checking against our physical reasoning in Step 2, we see that our answer makes sense.)

EXERCISE 2.3.02



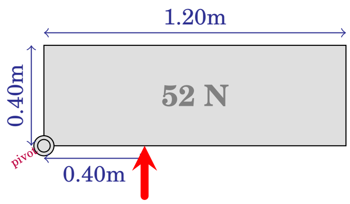
Answer: The applied force that maintains static equilibrium is 52.0 N, vertically upwards. The force at the pivot is zero since the applied force supports the entire weight.

EXERCISE 2.3.03



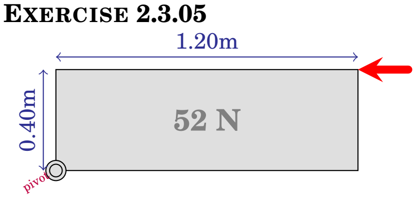
Answer: The applied force that maintains static equilibrium is 39.0 N, vertically upwards. The force at the pivot is 13.0 N, vertically upwards. Compare this result with the two previous exercises.

EXERCISE 2.3.04



Answer: The applied force that maintains static equilibrium is 78.0 N, vertically upwards. This is larger than the weight because it is closer to the pivot than the weight. The force at the pivot is 26.0 N, vertically *downwards*. Because the applied force is upwards greater than the weight the pivot has to supply a downward force to keep the object attached to the pivot.

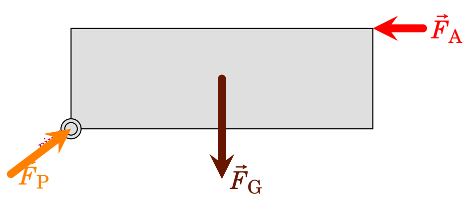
EXERCISE 2.3.05

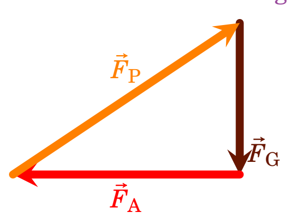


Step 0: The object is the box.

Step 1: There is gravity acting downwards at the center of the object, which exerts a clock-wise torque. And there is the applied force acting towards the left at the top right-hand corner, which applies a counter-clock-wise torque. There must be a force at the pivot because the applied force and gravity (being along different directions) can not cancel each other. The force at the pivot does not exert a torque.

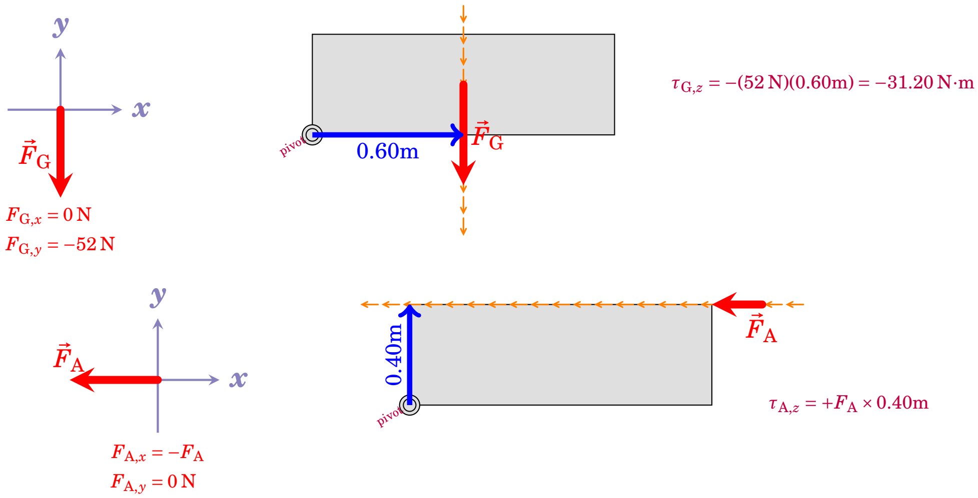
Step 2: The Free-Body Diagram, and the check of the sum of forces is below. Remember that we are only able to guess the force at the pivot on the FBD, and that is after we've tried to make the sum of forces diagram.





Gravity is vertically downwards and the applied force is horizontal towards the left. Since the applied force is closer to the pivot, to balance the torques we will expect that the applied force is larger (in magnitude) than gravity. This leads us to conclude that the force at the pivot will have to be towards the right and slightly upwards.

Step 3: The components of the forces, and the contributions to the torque about the pivot are determined below. With gravity and the applied force being vertical and horizontal, respectively, the moment arms are clearly visible on the rectangular object, and we can use "$\tau_z = \pm F d$" formula to calculate each torque.



The components of the force at the pivot ($F_{P,x}$ and $F_{P,y}$) are both unknowns. This force exerts no torque ($\tau_{P,z} = 0\,N{\cdot}m$) since it is acting *at* the pivot (its moment arm is $d = 0\,m$).

Step 4: The conditions for static equilibrium given by Newton's 1st Law are

$$F_{G,x} + F_{A,x} + F_{P,x} = 0\,N \qquad 0\,N + (-F_A) + F_{P,x} = 0\,N \tag{2.7}$$
$$F_{G,y} + F_{A,y} + F_{P,y} = 0\,N \qquad (-52\,N) + (0\,N) + F_{P,y} = 0\,N \tag{2.8}$$
$$\tau_{G,z} + \tau_{A,z} + \tau_{P,z} = 0\,N{\cdot}m \qquad (-31.20\,N{\cdot}m) + (+F_A \times 0.40m) + (0\,N{\cdot}m) = 0\,N{\cdot}m \tag{2.9}$$

Step 5: Solving the torque equation we find that $F_A = 78.0\,N$. Substituting this into the equation for the sum of the x-components, we see that $F_{P,x} = +78.0\,N$. Solving the equation for the sum of the y-components, we find that $F_{P,y} = +52.0\,N$.

The force at the pivot has magnitude

$$F_P = \sqrt{F_{P,x}^2 + F_{P,y}^2} = \sqrt{(+78.0\,N)^2 + (+52.0\,N)^2} = 93.7\,N \tag{2.10}$$

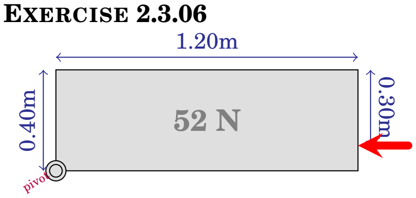
The vector $\vec{F}_P$ points into the first quadrant, making an angle

$$\theta = \tan^{-1}\left(\frac{F_{P,y}}{F_{P,x}}\right) = \tan^{-1}\left(\frac{+52.0\,N}{+78.0\,N}\right) = 33.7° \tag{2.11}$$

above the $+x$-axis.

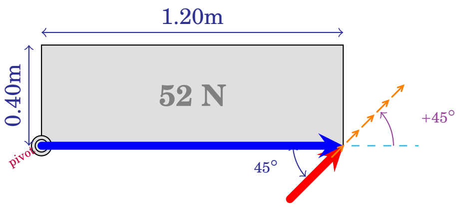
Answer: The applied force that maintains static equilibrium is 78 N, towards the left. This is larger than the weight because its line of action is closer to the pivot than the weight's. The force at the pivot is 94 N, to the right and slightly upwards at an angle of 34° above the $+x$-direction. Because the applied force is towards the left and gravity is downwards the pivot has to supply a force that balances both these components.

Exercise 2.3.06



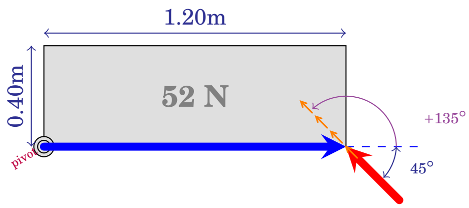
Answer: The applied force that maintains static equilibrium is 312 N, towards the left. This is much larger than the weight because its line of action is so much closer to the pivot than the weight's. The force at the pivot is 316 N, to the right and very slightly upwards at an angle of 9.5° above the $+x$-direction. Because the applied force is towards the left and gravity is downwards the pivot has to supply a force that balances both these components.

Exercise 2.3.07



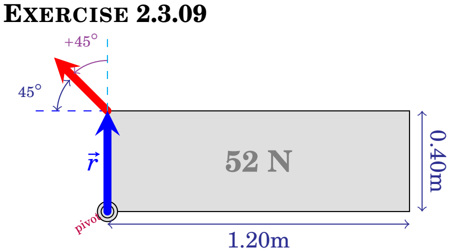
Answer: The applied force that maintains static equilibrium is 37 N, upwards and to the right at 45° above the $+x$-axis. This is a little more than half the weight because its perpendicular component ($F_\perp$) is half the weight. The force at the pivot is 37 N, upwards and to the left at 45° above the $-x$-axis. Because the applied force has a portion towards the right and gravity is downwards the pivot has to supply a force that balances both these components.

Exercise 2.3.08



Answer: The applied force that maintains static equilibrium is 37 N, upwards and to the left at 45° above the $-x$-axis. This is a little more than half the weight because its perpendicular component ($F_\perp$) is half the weight. The force at the pivot is 37 N, upwards and to the right at 45° above the $+x$-axis. Because the applied force has a portion towards the left and gravity is downwards the pivot has to supply a force that balances both these components.

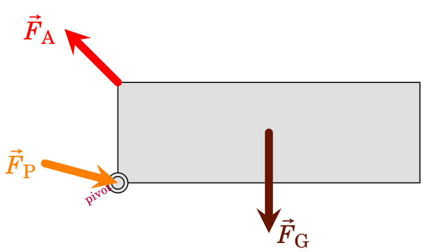
Exercise 2.3.09

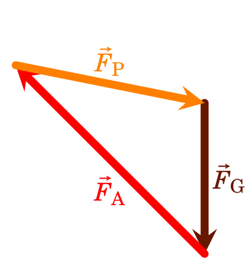


Step 0: The object is the box.

Step 1: There is gravity acting downwards at the center of the object, which exerts a clock-wise torque. And there is the applied force acting towards the left at the top left-hand corner, which applies a counter-clockwise torque. There must be a force at the pivot because the applied force and gravity (being along different directions) can not cancel each other. The force at the pivot does not exert a torque.

Step 2: The Free-Body Diagram, and the check of the sum of forces is below. Remember that we are only able to guess the force at the pivot on the FBD, and that is after we've tried to make the sum of forces diagram.

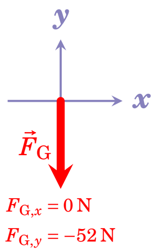


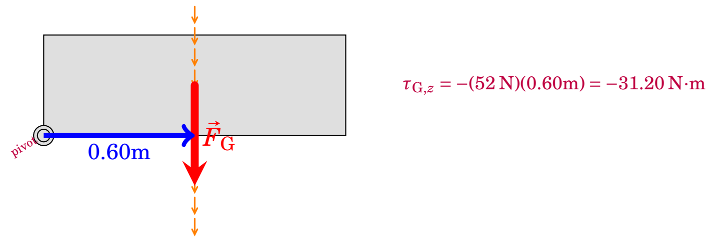


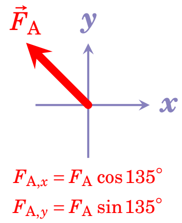
Gravity is vertically downwards and the applied force is angled upwards towards the left. Since the applied force is closer to the pivot, to balance the torques we will expect that the applied force is larger (in magnitude) than gravity. This leads us to conclude that the force at the pivot will have to point towards the right, and may be angled slightly either upwards or downwards.

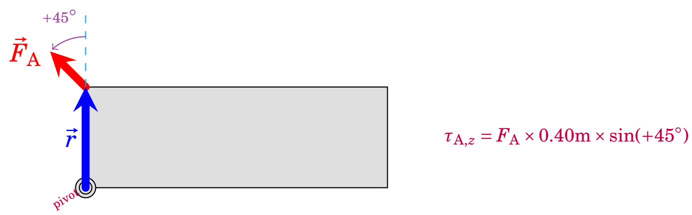
(*Advanced*: Since the applied force makes a 45° angle with the horizontal axes, its x and y components will have the same size. Since the vector $\vec{r}$ from the pivot to the applied force is small than the moment arm for gravity, perpendicular component $F_\perp$ of the applied force will have to be bigger than the weight. But $F_\perp$ is the same as the x-component of the applied force. This means that the force at the pivot will have to point downwards along with gravity to balance the vertical portion of the applied force.)

Step 3: The components of the forces, and the contributions to the torque about the pivot are determined below. With gravity the moment arm is clearly visible on the rectangular object, and we can use "$\tau_z = \pm F d$" formula to calculate its torque. For the applied force will have to use the "$\tau_z = F r \sin\theta$" formula to find its contribution to torque.









The components of the force at the pivot ($F_{\text{P},x}$ and $F_{\text{P},y}$) are both unknowns. This force exerts no torque ($\tau_{\text{P},z} = 0\,\text{N}\text{m}$) since it is acting *at* the pivot (its moment arm is $d = 0\,\text{m}$).

Step 4: The conditions for static equilibrium given by Newton's 1st Law are

$$F_{\text{G},x} + F_{\text{A},x} + F_{\text{P},x} = 0\,\text{N} \qquad 0\,\text{N} + (F_\text{A} \cos 135°) + F_{\text{P},x} = 0\,\text{N} \tag{2.12}$$

$$F_{\text{G},y} + F_{\text{A},y} + F_{\text{P},y} = 0\,\text{N} \qquad (-52\,\text{N}) + (F_\text{A} \sin 135°) + F_{\text{P},y} = 0\,\text{N} \tag{2.13}$$

$$\tau_{\text{G},z} + \tau_{\text{A},z} + \tau_{\text{P},z} = 0\,\text{N}\cdot\text{m} \qquad (-31.20\,\text{N}\cdot\text{m}) + \big(F_\text{A} \times 0.40\text{m} \times \sin(+45°)\big) + (0\,\text{N}\cdot\text{m}) = 0\,\text{N}\cdot\text{m} \tag{2.14}$$

Step 5: Solving the torque equation we find that the magnitude of the applied force is $F_\text{A} = 110.3\,\text{N}$ (which is larger than the weight, as expected). Substituting this into the equation for the sum of the x-components, we see that $F_{\text{P},x} = +78.0\,\text{N}$. Solving the equation for the sum of the y-components, we find that $F_{\text{P},y} = -26.0\,\text{N}$.

The force at the pivot has magnitude

$$F_\text{P} = \sqrt{F_{\text{P},x}^2 + F_{\text{P},y}^2} = \sqrt{(+78.0\,\text{N})^2 + (-26.0\,\text{N})^2} = 82.2\,\text{N} \tag{2.15}$$

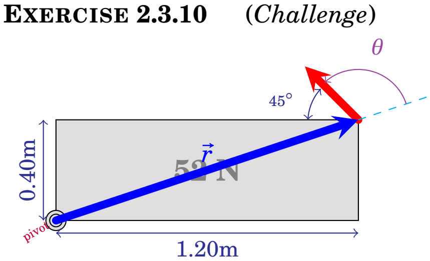
The vector $\vec{F}_\text{P}$ points into the fourth quadrant (below the $+x$-axis), making an angle

$$\theta = \tan^{-1}\left(\frac{F_{\text{P},y}}{F_{\text{P},x}}\right) = \tan^{-1}\left(\frac{-26.0\,\text{N}}{+78.0\,\text{N}}\right) = -18.4° \tag{2.16}$$

below the $+x$-axis.

Answer: The applied force that maintains static equilibrium is 110 N, to the left and upwards at 45°. This is larger than the weight because its line of action is closer to the pivot than the weight's. The force at the pivot is 82.2 N, to the right and slightly downwards at an angle of 18° below the $+x$-direction.

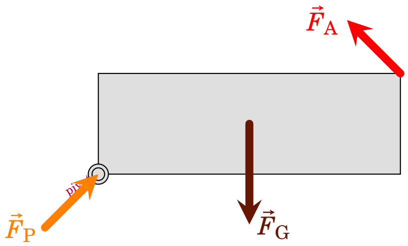
EXERCISE 2.3.10 (*Challenge*)

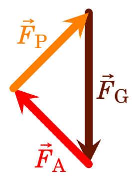


Step 0: The object is the box.

Step 1: There is gravity acting downwards at the center of the object, which exerts a clock-wise torque. And there is the applied force acting towards the left at the top right-hand corner, which applies a counter-clockwise torque. There must be a force at the pivot because the applied force and gravity (being along different directions) can not cancel each other. The force at the pivot does not exert a torque.

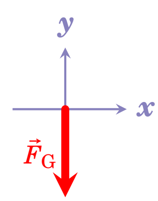
Step 2: The Free-Body Diagram, and the check of the sum of forces is below. Remember that we are only able to guess the force at the pivot on the FBD, and that is after we've tried to make the sum of forces diagram.

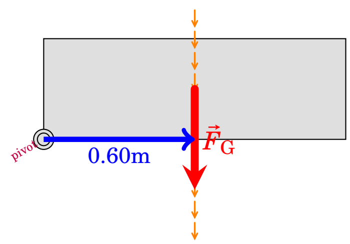




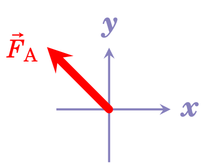
Gravity is vertically downwards and the applied force is angled upwards towards the left. Since the applied force is further from the pivot, to balance the torques we will expect that the applied force is smaller (in magnitude) than gravity. This leads us to conclude that the force at the pivot will have to point towards the right, and is probably angled upwards.

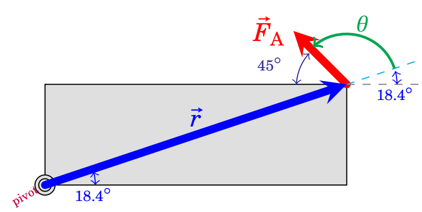
Step 3: The components of the forces, and the contributions to the torque about the pivot are determined below. With gravity the moment arm is clearly visible on the rectangular object, and we can use "$\tau_z = \pm F d$" formula to calculate its torque. For the applied force will have to use the "$\tau_z = F r \sin\theta$" formula to find its contribution to torque.





$\tau_{G,z} = -(52\,\text{N})(0.60\text{m}) = -31.20\,\text{N}\cdot\text{m}$





$r_x = +1.20\,\text{m}$
$r_y = +0.40\,\text{m}$
$r = \sqrt{r_x^2 + r_y^2} = 1.265\,\text{m}$
$\tan^{-1}\left(\frac{+0.40\,\text{m}}{+1.20\,\text{m}}\right) = 18.4^\circ$

$\tau_{A,z} = F_A \times 1.265\text{m} \times \sin(+135^\circ - 18.4^\circ)$
$= +F_A \times 1.131\text{m}$

The length of the radial vector $\vec{r}$ is 1.265 m. The angle between the horizontal and $\vec{r}$ is 18.4°. Thus the angle between the radial vector and the applied force is $\theta = +135^\circ - 18.4^\circ = 116.6^\circ$.

The components of the force at the pivot ($F_{P,x}$ and $F_{P,y}$) are both unknowns. This force exerts no torque ($\tau_{P,z} = 0\,\text{N}\text{m}$) since it is acting *at* the pivot (its moment arm is $d = 0\,\text{m}$).

Step 4: The conditions for static equilibrium given by Newton's 1st Law are

$$F_{G,x} + F_{A,x} + F_{P,x} = 0\,\text{N} \qquad 0\,\text{N} + (F_A \cos 135^\circ) + F_{P,x} = 0\,\text{N} \tag{2.17}$$
$$F_{G,y} + F_{A,y} + F_{P,y} = 0\,\text{N} \qquad (-52\,\text{N}) + (F_A \sin 135^\circ) + F_{P,y} = 0\,\text{N} \tag{2.18}$$
$$\tau_{G,z} + \tau_{A,z} + \tau_{P,z} = 0\,\text{N}\cdot\text{m} \qquad (-31.20\,\text{N}\cdot\text{m}) + (F_A \times 1.131\text{m}) + (0\,\text{N}\cdot\text{m}) = 0\,\text{N}\cdot\text{m} \tag{2.19}$$

Step 5: Solving the torque equation we find that the magnitude of the applied force is $F_A = 27.6\,\text{N}$ (which is smaller than the weight, as expected). Substituting this into the equation for the sum of the x-components, we see that $F_{P,x} = +19.5\,\text{N}$. Solving the equation for the sum of the y-components, we find that $F_{P,y} = +32.5\,\text{N}$.

The force at the pivot has magnitude

$$F_{\mathrm{P}} = \sqrt{F_{\mathrm{P},x}^2 + F_{\mathrm{P},y}^2} = \sqrt{(+19.5\,\mathrm{N})^2 + (+32.5\,\mathrm{N})^2} = 37.9\,\mathrm{N} \tag{2.20}$$

The vector $\vec{F}_{\mathrm{P}}$ points into the first quadrant, making an angle

$$\theta = \tan^{-1}\left(\frac{F_{\mathrm{P},y}}{F_{\mathrm{P},x}}\right) = \tan^{-1}\left(\frac{+32.5\,\mathrm{N}}{+19.5\,\mathrm{N}}\right) = 59.0^\circ \tag{2.21}$$

above the $+x$-axis.

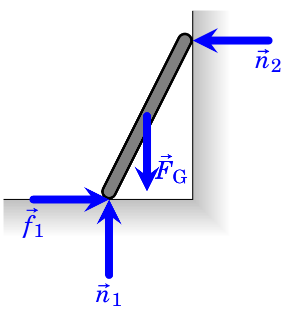
Answer: The applied force that maintains static equilibrium is 28 N, to the left and upwards at 45°. This is smaller than the weight because its line of action is further from the pivot than the weight's. The force at the pivot is 38 N, upwards to the right at an angle of 59° above the $+x$-direction.

Rods, Ropes & Surfaces

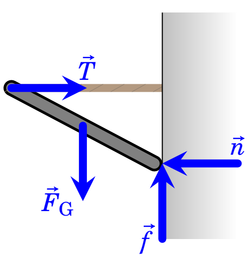
In each of the following situations, solve for each of the unknown forces. Find the normal at each surface of contact. In the cases when it is non-zero, find the magnitude and direction of friction. Find the tension in each rope that is present.

A hint on how to start problems of this type: Chose a point of contact with a surface as the pivot, preferably one where there is friction (when it is present). This will eliminate one (perhaps two!) unknowns from the torque equation, simplifying the algebra.

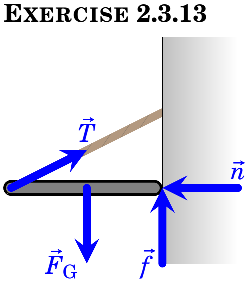
EXERCISE 2.3.11



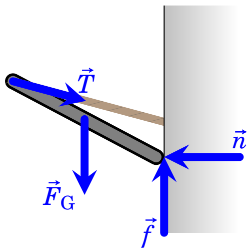
EXERCISE 2.3.12



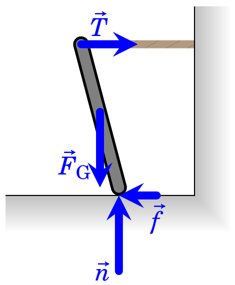
EXERCISE 2.3.13



EXERCISE 2.3.14



EXERCISE 2.3.15

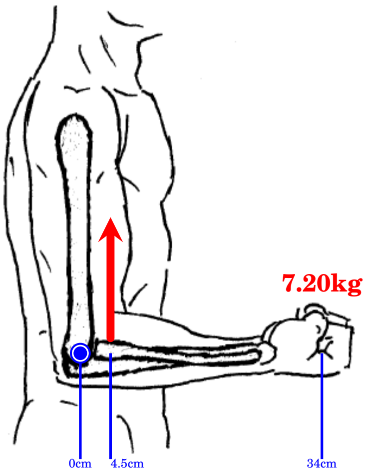


2.3.2 Biomechanical Equilibrium

In all the problems that follow, after finding the unknown force, find the force exerted at the pivot.

PROBLEM 2.3.01:

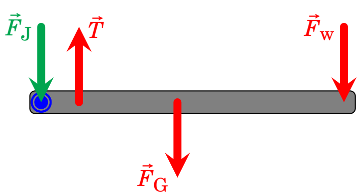
A person is holding a 7.20 kg mass in their hand. What total amount of force must the muscles of their upper arm be exerting on their lower arm? The pivot in this situation is the elbow joint. The active muscles are the biceps and brachialis, which are the large muscles on the front side of the upper arm. They attach to the forearm 4.5 cm from the elbow joint, as shown in the diagram. The forearm (including the hand) has a mass of 1.80 kg, and has its center of mass 15.0 cm from the elbow joint.



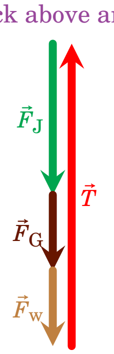
Step 0: The object is the person's forearm including the hand.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the forearm ($\vec{F}_{\mathrm{G}}$). The mass in the person's hand is pressing downwards ($\vec{F}_{\mathrm{w}}$). The tension in the muscles of the upper arm ($\vec{T}$) are pulling upwards on the forearm, which bring the bones into contact with each other. Thus there will be a force *at* the joint ($\vec{F}_{\mathrm{J}}$).

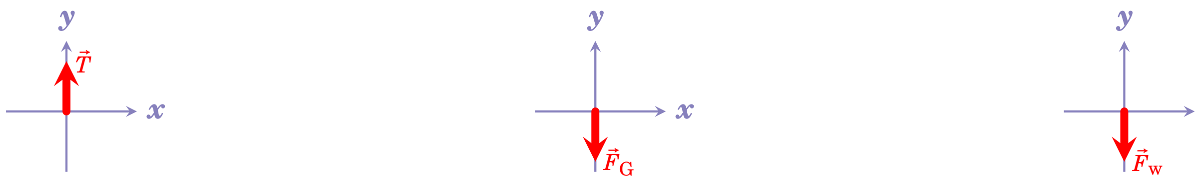
Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):



With all the forces being vertical, two of which are unknown, we can not (yet) construct a check of the sum of forces. But we do know that the tension is almost eight times closer to the pivot than the weight in the hand. For this we expect that the tension will about eight times larger than the weight in the hand. With this we can sketch the sum (below). From this it is our guess that the contact force at the joint ($\vec{F}_{\mathrm{J}}$) will be large and pointed downwards. (We go back above and add that to our FBD.)



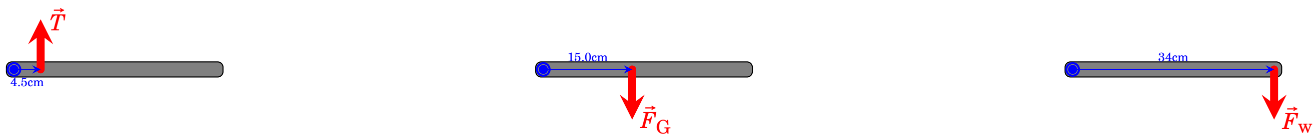
Step 3: All the forces acting in the situation are vertical, so all x-components are zero. The components of these forces, and their contributions to the torque about the pivot, are:



$T_x = 0\,\mathrm{N}$
$T_y = +T$

$F_{\mathrm{G},y} = 0\,\mathrm{N}$
$F_{\mathrm{G},y} = -(9.81\,\frac{\mathrm{N}}{\mathrm{kg}}) \times 1.80\mathrm{kg} = -17.658\,\mathrm{N}$

$F_{\mathrm{G},w} = 0\,\mathrm{N}$
$F_{\mathrm{w},y} = -9.81\,\frac{\mathrm{N}}{\mathrm{kg}} \times 7.20\mathrm{kg} = -70.632\,\mathrm{N}$



$\tau_{\mathrm{T},z} = +0.045\mathrm{m} \times T$

$\tau_{\mathrm{G},z} = -0.150\mathrm{m} \times 9.81\,\frac{\mathrm{N}}{\mathrm{kg}} \times 1.80\mathrm{kg}$
$= -2.649\,\mathrm{N{\cdot}m}$

$\tau_{\mathrm{w},z} = -0.340\mathrm{m} \times 9.81\,\frac{\mathrm{N}}{\mathrm{kg}} \times 7.20\mathrm{kg}$
$= -24.015\,\mathrm{N{\cdot}m}$

The components of $\vec{F}_{\mathrm{J}}$ are unknowns to be solved. The torque due to the force at the pivot ($\vec{F}_{\mathrm{J}}$) is just $0\,\mathrm{N{\cdot}m}$.

Step 4: With the arm in static equilibrium, the torques acting on the arm must sum to zero:

$$\tau_{\mathrm{T},z} + \tau_{\mathrm{G},z} + \tau_{\mathrm{w},z} + \tau_{\mathrm{J},z} = 0\,\mathrm{N{\cdot}m} \tag{2.22}$$
$$(+0.045\mathrm{m} \times T) - 2.649\,\mathrm{N{\cdot}m} - 24.015\,\mathrm{N{\cdot}m} + 0\,\mathrm{N{\cdot}m} = 0\,\mathrm{N{\cdot}m} \tag{2.23}$$

(As noted previously, there is no contribution to torque from the force at the joint since the moment arm of that force is zero.) All horizontal components of force are zero, identically. All vertical components of force must sum to zero:

$$F_{\mathrm{J},y} + T_y + F_{\mathrm{G},y} + F_{\mathrm{w},y} = 0\,\mathrm{N} \tag{2.24}$$
$$F_{\mathrm{J},y} + T - 17.658\,\mathrm{N} - 70.632\,\mathrm{N} = 0\,\mathrm{N} \tag{2.25}$$

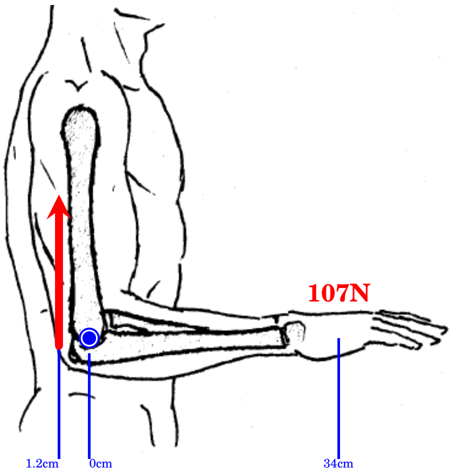
Step 5: Solving the sum-of-torques equation gives us that $T = 593\,\mathrm{N}$, which points upwards (the direction was given). Using that result to solve the sum-of-forces equation gives us that the vertical component of the force at the joint is $F_{\mathrm{J},y} = -504\,\mathrm{N}$.

Answer: The magnitude of the force exerted by the muscles is 593 N. The force exerted at the joint is 504 N, downwards.

Why is the force at the joint downwards? Since the muscles exert a force much closer to the joint than gravity or the weight, and its torque must balance, it will be a much larger force than either. To balance this large upwards force, there must be a similarly large downwards force at the joint.

Problem 2.3.02:

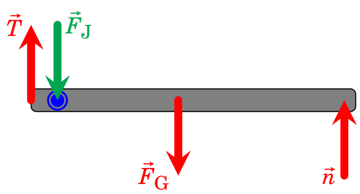
A person is pressing downwards with their hand, exerting 107 N on a surface (not shown). What total amount of force must the muscles of their upper arm be exerting on their lower arm? (The active muscles are the cluster referred to as the triceps, which are the muscles on the rear side of the upper arm.) The forearm (including the hand) has a mass of 1.8kg, and has its center of mass 15cm from the elbow joint.



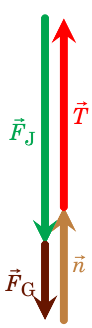
Step 0: The object is the person's forearm including the hand.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the forearm. The is pressing downwards on a surface, and the reaction to that is the surface pushing upwards against the person. The muscles of the upper arm are pulling upwards on the forearm, which bring the bones into contact with each other.

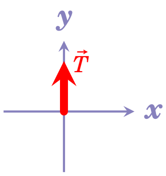
Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):



In this system the tension is almost *thirty* times closer to the pivot than the force at the hand! For this we expect that the tension will about thirty times larger than the force at the hand. With this we can sketch the sum (below). From this it is our guess that the contact force at the joint ($\vec{F}_J$) will be very large and pointed downwards. (We go back above and add that to our FBD.)

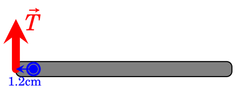


Step 3: All the forces acting in the situation are vertical, so all x-components are zero. The components of these forces, and their contributions to the torque about the pivot, are:

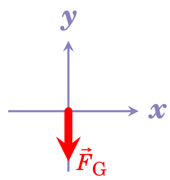


$T_x = 0\,\mathrm{N}$

$T_y = +T$

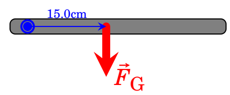


$\tau_{\mathrm{T},z} = -0.012\mathrm{m} \times T$



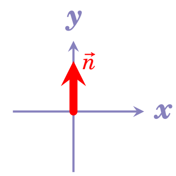
$F_{\mathrm{G},y} = 0\,\mathrm{N}$

$F_{\mathrm{G},y} = -(9.81\,\frac{\mathrm{N}}{\mathrm{kg}}) \times 1.80\mathrm{kg} = -17.658\,\mathrm{N}$



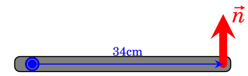
$\tau_{\mathrm{G},z} = -0.150\mathrm{m} \times 9.81\,\frac{\mathrm{N}}{\mathrm{kg}} \times 1.80\mathrm{kg}$

$= -2.649\,\mathrm{N{\cdot}m}$



$n_x = 0\,\mathrm{N}$

$n_y = +107\,\mathrm{N}$



$\tau_{\mathrm{n},z} = +0.340\mathrm{m} \times 107\mathrm{N}$

$= +36.380\,\mathrm{N{\cdot}m}$

The components of $\vec{F}_{\mathrm{J}}$ are unknowns to be solved. The torque due to the force at the pivot ($\vec{F}_{\mathrm{J}}$) is just $0\,\mathrm{N{\cdot}m}$.

Step 4: With the arm in static equilibrium, the torques acting on the arm must sum to zero:

$$\tau_{\mathrm{T},z} + \tau_{\mathrm{G},z} + \tau_{\mathrm{n},z} + \tau_{\mathrm{J},z} = 0\,\mathrm{N{\cdot}m} \tag{2.26}$$

$$(-0.012\mathrm{m} \times T) - 2.649\,\mathrm{N{\cdot}m} + 36.380\,\mathrm{N{\cdot}m} + 0\,\mathrm{N{\cdot}m} = 0\,\mathrm{N{\cdot}m} \tag{2.27}$$

(As noted previously, there is no contribution to torque from the force at the joint since the moment arm of that force is zero.) All horizontal components of force are zero, identically. All vertical components of force must sum to zero:

$$T_y + F_{\mathrm{J},y} + F_{\mathrm{G},y} + n_y = 0\,\mathrm{N} \tag{2.28}$$

$$+T + F_{\mathrm{J},y} - 17.658\,\mathrm{N} + 107\,\mathrm{N} = 0\,\mathrm{N} \tag{2.29}$$

Step 5: Solving the sum-of-torques equation gives us that $T = 2811\,\mathrm{N}$, which points upwards (the direction was given). Using that result to solve the sum-of-forces equation gives us that the vertical component of the force at the joint is $F_{\mathrm{J},y} = -2900\,\mathrm{N}$.

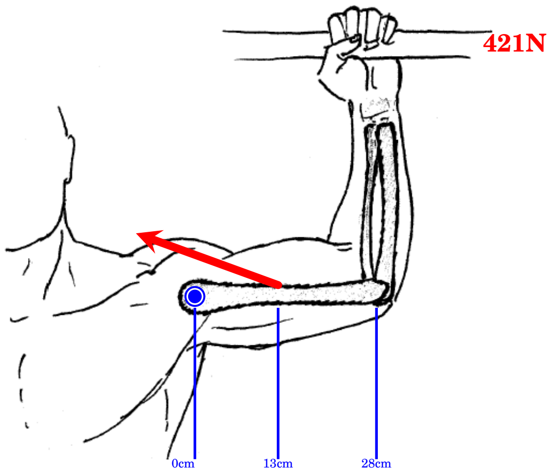
Answer: The magnitude of the force exerted by the muscles is 2811 N. The force exerted at the joint is 2900 N, downwards.

Why is the force at the joint downwards? Since the muscles exert a force much closer to the joint than the normal, and its torque must balance, it will be a much larger force. To balance this large upwards force, there must be a similarly large downwards force at the joint.

We pause to note that the forces exerted by the muscle and exerted in the joint are almost thirty times larger than those exerted by gravity or exerted at the extremity of the limb. This is a typical situation!

PROBLEM 2.3.03:

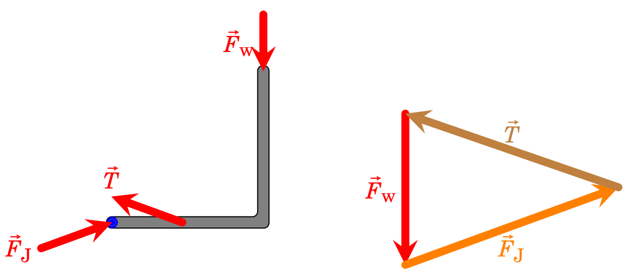
A weightlifter is holding a weight above their head. Each arm supports 421N. The muscles of the shoulder collectively exert a force along a line that is 20° above the axis of the bone of the upper arm (the humerus). What is the magnitude of that collective force? (We will neglect the mass of the arm itself in this.)



Step 0: The object is person's entire arm, including the hand.

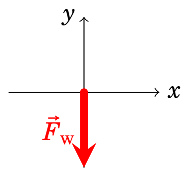
Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the arm, but we are told to ignore that contribution (presumably because it will be much smaller than the other forces in the situation). The person is holding the weights above their head, so the weights push downwards on the person's hand. The muscles in the shoulder (primarily the deltoids) pull on the arm at an angle, pressing the bones of the arm (specifically the humerus) into contact with the bones of the shoulder (specifically the scapula).

Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):

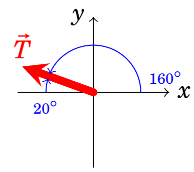


The force at the joint may have to point slightly downwards if the force due to the muscle is very large in comparison to the weight being supported.

Step 3: The components of the forces acting are



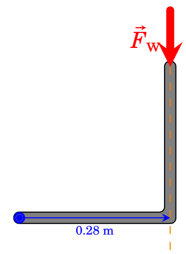
$F_{\text{w},x} = 0\,\text{N}$
$F_{\text{w},y} = -421\,\text{N}$



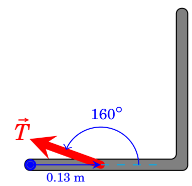
$T_x = T\cos 160^\circ$
$T_y = T\sin 160^\circ$

While we expect that the force at the joint will point towards the right, its components remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:



$\tau_{\text{w}} = -(421\text{N})(0.28\text{m})$
$= -117.9\,\text{N}\cdot\text{m}$



$\tau_T = +(T)(0.13\text{m})\sin 160^\circ$
$= +T \times 0.04446\text{m}$

Notice how, in the case of the force due to the weight, we were able to use the line of action to find the moment arm directly, while, in the case of the force due to the muscle, we needed to find the distance to point where the force was acting and the angle to the line of action.

As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_{\text{J}} + \vec{T} + \vec{F}_{\text{w}} = \vec{0}\,\text{N} \tag{2.30}$$

The x-component of the sum is

$$F_{\text{J},x} + T_x + F_{\text{w},x} = 0\,\text{N} \tag{2.31}$$
$$F_{\text{J},x} + T\cos 160^\circ + 0\,\text{N} = 0\,\text{N} \tag{2.32}$$

The y-component of the sum is

$$F_{\text{J},y} + T_y + F_{\text{w},y} = 0\,\text{N} \tag{2.33}$$
$$F_{\text{J},y} + T\sin 160^\circ - 421\,\text{N} = 0\,\text{N} \tag{2.34}$$

The sum of torques must also be zero:

$$\tau_T + \tau_{\text{w}} = 0\,\text{N}\cdot\text{m} \tag{2.35}$$
$$(+T \times 0.04446\text{m}) - 117.9\,\text{N}\cdot\text{m} = 0\,\text{N}\cdot\text{m} \tag{2.36}$$

Step 5: Solving the sum-of-torques equation gives us $T = 2651\,\text{N}$. Using this in the sum-of-forces equations gives

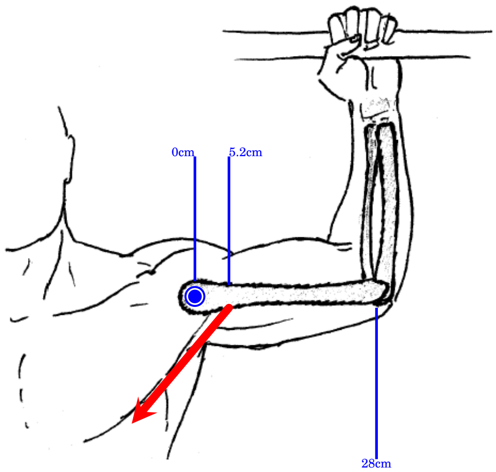
$$F_{\text{J},x} = +2491\,\text{N} \tag{2.37}$$
$$F_{\text{J},y} = -486\,\text{N} \tag{2.38}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 2651\,\text{N}$. (As expected, since the muscle acts at a point much closer to the joint, the force applied by the muscle is much greater in magnitude than that of the weight.) The force at the joint has magnitude $F_J = 2538\,\text{N}$, and points towards the right and 11° below the horizontal. (As noted when we checked the sum-of-forces, because the force exerted by the muscles is so large the contact force between the bones at the joint must point slightly downwards to counter the large upwards component from the muscles.)

We can end with a note that the weight of the arm (which would have been about 20 N) would have contributed a few percent to the forces already present. Ignoring the weight did not affect the results significantly.

PROBLEM 2.3.04:

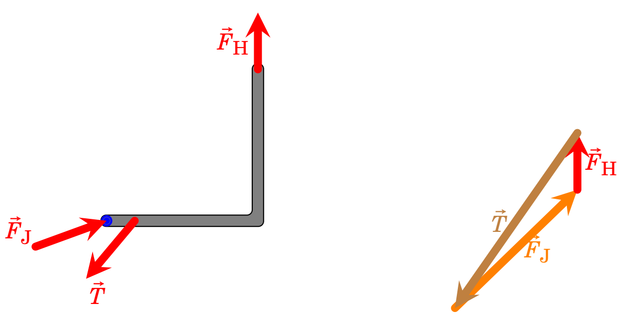
A person is doing a pull-up. Each arm support half their mass (total mass 70kg). The pectorals and the muscles of the back collectively exert a force along a line that is 40° to the left of the vertical. What is the magnitude of that collective force? (We will neglect the mass of the arm itself in this.)



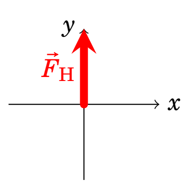
Step 0: The object is person's entire arm, including the hand.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the arm, but we are told to ignore that contribution (presumably because it will be much smaller than the other forces in the situation). The person is holding their own weight by their hands, so the bar they are holding must be pulling upwards on their hands. The muscles of the torso (primarily the pectorals on the front, and the latissimus dorsi on the back) pull on the arm downwards at an angle, pressing the bones at the joint into contact.

Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):

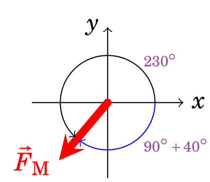


Step 3: The components of the forces acting are



$F_{H,x} = 0\,\mathrm{N}$

$F_{H,y} = (\frac{1}{2} \times 70\mathrm{kg})(9.81\frac{\mathrm{N}}{\mathrm{kg}}) = +343.4\,\mathrm{N}$

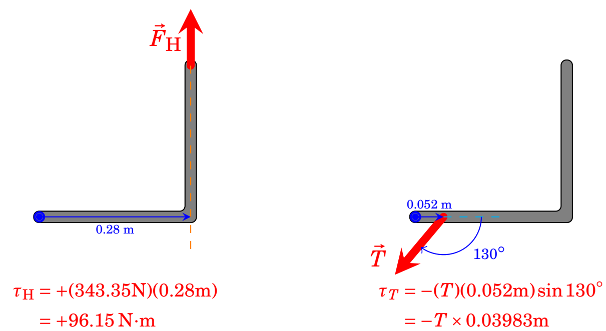


$T_x = T\cos 230°$

$T_y = T\sin 230°$

The components of the force at the joint remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:



As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_\mathrm{J} + \vec{T} + \vec{F}_\mathrm{H} = \vec{0}\,\mathrm{N} \tag{2.39}$$

The x-component of the sum is

$$F_{\mathrm{J},x} + T_x + F_{\mathrm{H},x} = 0\,\mathrm{N} \tag{2.40}$$
$$F_{\mathrm{J},x} + T\cos 230^\circ + 0\,\mathrm{N} = 0\,\mathrm{N} \tag{2.41}$$

The y-component of the sum is

$$F_{\mathrm{J},y} + T_y + F_{\mathrm{H},y} = 0\,\mathrm{N} \tag{2.42}$$
$$F_{\mathrm{J},y} + T\sin 230^\circ + 343.4\,\mathrm{N} = 0\,\mathrm{N} \tag{2.43}$$

The sum of torques must also be zero:

$$\tau_T + \tau_\mathrm{H} = 0\,\mathrm{N{\cdot}m} \tag{2.44}$$
$$(-T \times 0.03983\mathrm{m}) + 96.15\,\mathrm{N{\cdot}m} = 0\,\mathrm{N{\cdot}m} \tag{2.45}$$

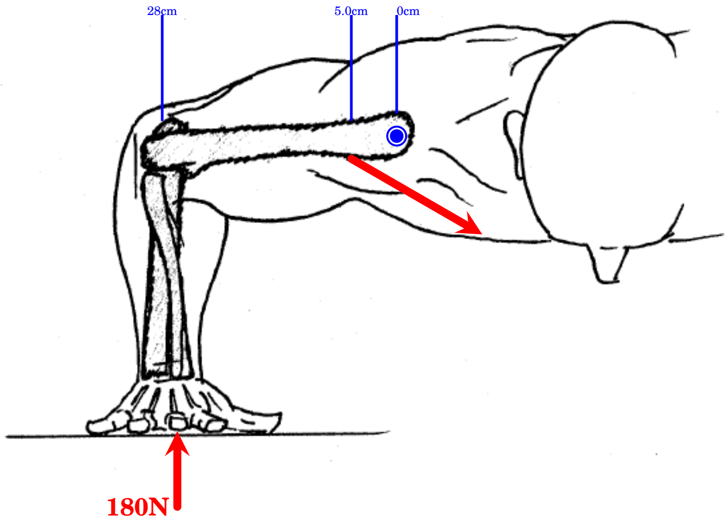
Step 5: Solving the sum-of-torques equation gives us $T = 2414\,\mathrm{N}$. Using this in the sum-of-forces equations gives

$$F_{\mathrm{J},x} = +1552\,\mathrm{N} \tag{2.46}$$
$$F_{\mathrm{J},y} = +1506\,\mathrm{N} \tag{2.47}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 2414\,\mathrm{N}$. The force at the joint has magnitude $F_\mathrm{J} = 2163\,\mathrm{N}$, and points towards the right and 44° above the horizontal. As always the forces generated at and around the joint are much larger than the forces at the extremities that are their cause.

PROBLEM 2.3.05:

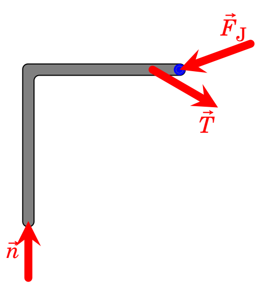
A person doing a push-up supports 180N at each hand. The pectoral muscle (the big muscle on the chest) attaches at an angle of 30° to the axis of the bone of the upper arm (the humerus). What is the magnitude of the force exerted by the pectoral? (We will neglect the mass of the arm in this.)

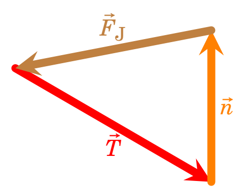


Step 0: The object is person's entire arm, including the hand.

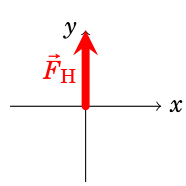
Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the arm, but we are told to ignore that contribution (presumably because it will be much smaller than the other forces in the situation). The person is touching the surface, supporting a portion of their own weight by their hands (the upwards force of 180N). The muscles on the chest (the pectorals) pull on the arm downwards at an angle, pressing the bones at the joint into contact.

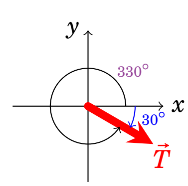
Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):





Step 3: The components of the forces acting are



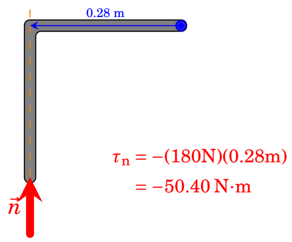


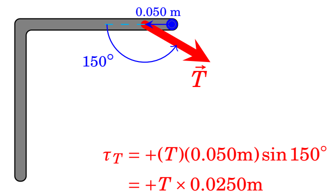
$n_x = 0\,\text{N}$
$n_y = +180\,\text{N}$

$T_x = T\cos 330°$
$T_y = T\sin 330°$

The components of the force at the joint remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:





As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_\text{J} + \vec{T} + \vec{n} = \vec{0}\,\text{N} \tag{2.48}$$

The x-component of the sum is

$$F_{\text{J},x} + T_x + n_x = 0\,\text{N} \tag{2.49}$$
$$F_{\text{J},x} + T\cos 330^\circ + 0\,\text{N} = 0\,\text{N} \tag{2.50}$$

The y-component of the sum is

$$F_{\text{J},y} + T_y + n_y = 0\,\text{N} \tag{2.51}$$
$$F_{\text{J},y} + T\sin 330^\circ + 180\,\text{N} = 0\,\text{N} \tag{2.52}$$

The sum of torques must also be zero:

$$\tau_T + \tau_\text{n} = 0\,\text{N}\cdot\text{m} \tag{2.53}$$
$$(+T \times 0.0250\text{m}) - 50.40\,\text{N}\cdot\text{m} = 0\,\text{N}\cdot\text{m} \tag{2.54}$$

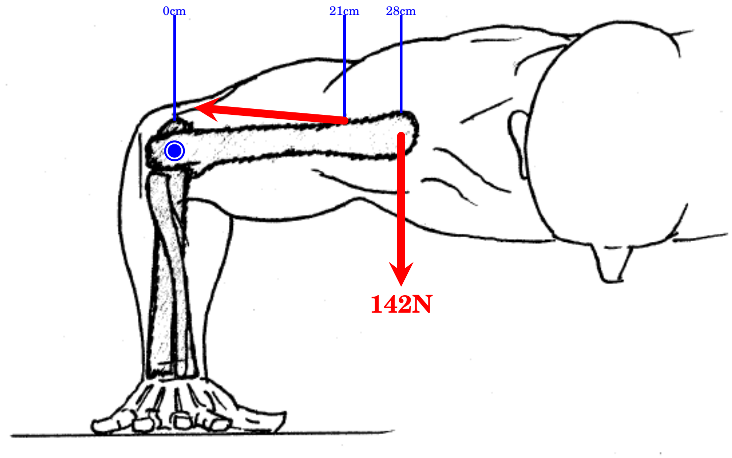
Step 5: Solving the sum-of-torques equation gives us $T = 2016\,\text{N}$. Using this in the sum-of-forces equations gives

$$F_{\text{J},x} = -1746\,\text{N} \tag{2.55}$$
$$F_{\text{J},y} = +828\,\text{N} \tag{2.56}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 2016\,\text{N}$. The force at the joint has magnitude $F_\text{J} = 1932\,\text{N}$, and points upwards and to the left at 155° counter-clockwise from the $+x$-axis.

PROBLEM 2.3.06:

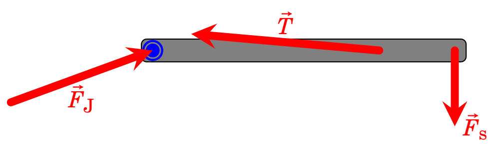
A person doing a push-up supports 142N at each shoulder joint. The triceps muscles (the muscles on the back of the upper arm) attaches at an angle of 5° to the axis of the bone of the upper arm (the humerus). What is the magnitude of the force exerted by the triceps? (We will neglect the mass of the arm in this.)

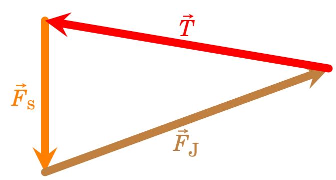


Step 0: The object is person's upper arm, from the elbow to the shoulder.

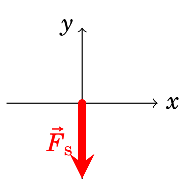
Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull on the arm, but we are told to ignore that contribution (presumably because it will be much smaller than the other forces in the situation). The muscles on the back of the upper arm (the triceps) pull on the arm towards the left at a very small angle angle, pressing the bones at the elbow joint into contact. The connection at the shoulder joint is transferring a portion of the person's weight (the 142N force) to the object.

Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):

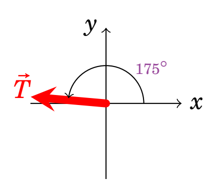




Step 3: The components of the forces acting are



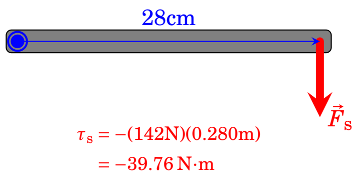
$F_{s,x} = 0\,\text{N}$
$F_{s,y} = -142\,\text{N}$

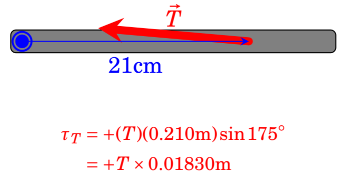


$T_x = T\cos 175°$
$T_y = T\sin 175°$

The components of the force at the joint remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:





As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_\mathrm{J} + \vec{T} + \vec{F}_\mathrm{s} = \vec{0}\,\mathrm{N} \tag{2.57}$$

The x-component of the sum is

$$F_{\mathrm{J},x} + T_x + F_{\mathrm{s},x} = 0\,\mathrm{N} \tag{2.58}$$
$$F_{\mathrm{J},x} + T\cos 175^\circ + 0\,\mathrm{N} = 0\,\mathrm{N} \tag{2.59}$$

The y-component of the sum is

$$F_{\mathrm{J},y} + T_y + F_{\mathrm{s},x} = 0\,\mathrm{N} \tag{2.60}$$
$$F_{\mathrm{J},y} + T\sin 175^\circ - 142\,\mathrm{N} = 0\,\mathrm{N} \tag{2.61}$$

The sum of torques must also be zero:

$$\tau_T + \tau_\mathrm{s} = 0\,\mathrm{N{\cdot}m} \tag{2.62}$$
$$(+T \times 0.01830\mathrm{m}) - 39.76\,\mathrm{N{\cdot}m} = 0\,\mathrm{N{\cdot}m} \tag{2.63}$$

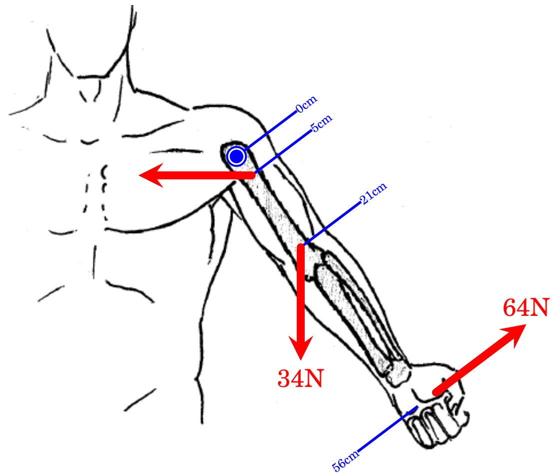
Step 5: Solving the sum-of-torques equation gives us $T = 2173\,\mathrm{N}$. Using this in the sum-of-forces equations gives

$$F_{\mathrm{J},x} = +2164\,\mathrm{N} \tag{2.64}$$
$$F_{\mathrm{J},y} = -47\,\mathrm{N} \tag{2.65}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 2173\,\mathrm{N}$. The force at the joint has magnitude $F_\mathrm{J} = 2165\,\mathrm{N}$, and points almost directly towards the right (only 1° below the $+x$-axis).

PROBLEM 2.3.07:

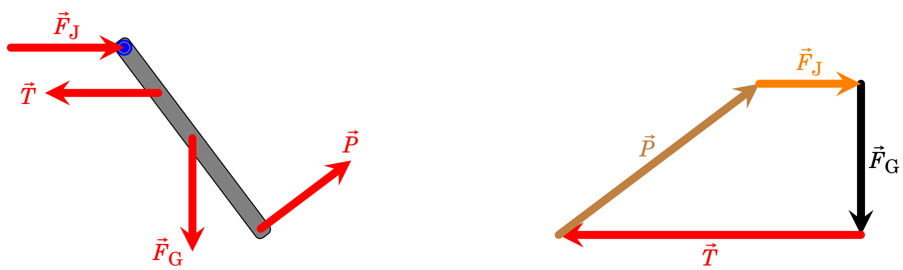
A person is holding a rope (not shown). The reaction force to their pull is the 64N force on their hand, perpendicular to their arm. The weight of their arm is 34N (center of mass 21cm from the shoulder). If their arm is 37° from the vertical, what force must their pectoral (chest muscle) exert horizontally?



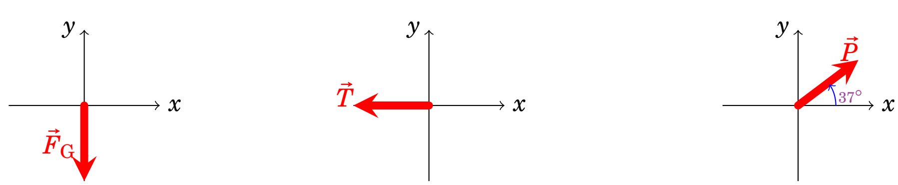
Step 0: The object is the person's whole arm.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull vertically downwards on the arm. The person is pulling on the rope (the action), and so the rope pulls on their fist (the reaction) upwards and to the right. The pectorals pull towards the left to balance the torque due to the force at the first. This presses the bones into contact in the shoulder joint.

Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):



Step 3: The components of the forces acting are



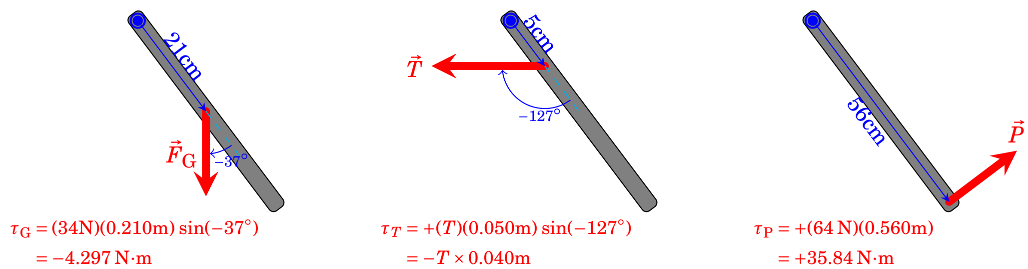
$F_{G,x} = 0\,\text{N}$
$F_{G,y} = -34\,\text{N}$

$T_x = -T$
$T_y = 0\,\text{N}$

$P_x = 64\,\text{N}\cos 37^\circ = +51.11\,\text{N}$
$P_y = 64\,\text{N}\sin 37^\circ = +38.52\,\text{N}$

The components of the force at the joint remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:



As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_G + \vec{T} + \vec{P} + \vec{F}_J = \vec{0}\,N \tag{2.66}$$

The x-component of the sum is

$$F_{G,x} + T_x + P_x + F_{J,x} = 0\,N \tag{2.67}$$
$$0\,N - T + 51.11\,N + F_{J,x} = 0\,N \tag{2.68}$$

The y-component of the sum is

$$F_{G,y} + T_y + P_y + F_{J,y} = 0\,N \tag{2.69}$$
$$-34.00\,N + 0\,N + 38.52\,N + F_{J,y} = 0\,N \tag{2.70}$$

The sum of torques must also be zero:

$$\tau_G + \tau_T + \tau_P + \tau_J = 0\,N\cdot m \tag{2.71}$$
$$-4.297\,N\cdot m - T \times 0.040m + 35.84\,N\cdot m + 0\,N\cdot m = 0\,N\cdot m \tag{2.72}$$

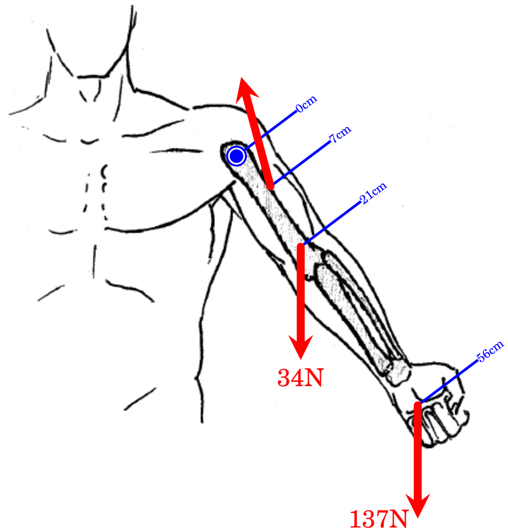
Step 5: Solving the sum-of-torques equation gives us $T = 790\,N$. Using this in the sum-of-forces equations gives

$$F_{J,x} = +739\,N \tag{2.73}$$
$$F_{J,y} = -5\,N \tag{2.74}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 790\,N$. The force at the joint has magnitude $F_J = 739\,N$, and points almost directly towards the right (less than 1° below the $+x$-axis).

PROBLEM 2.3.08:

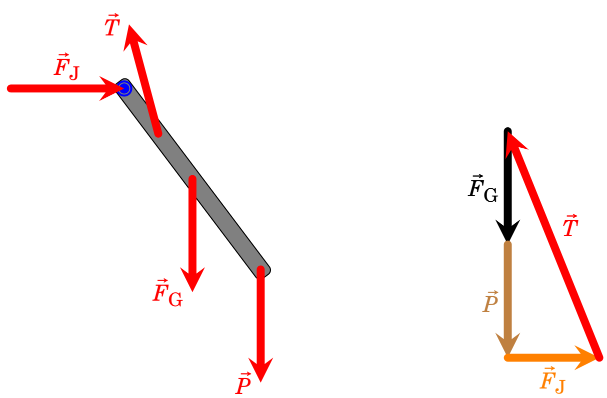
A person is lifting a 137N weight out to their side. The weight of their arm is 34N (center of mass 21cm from the shoulder). If their arm is 37° from the vertical, what force must their deltoid (shoulder muscle) exert? (The deltoid attaches to the humerus 7.0cm from the shoulder joint at an angle of 22°.)



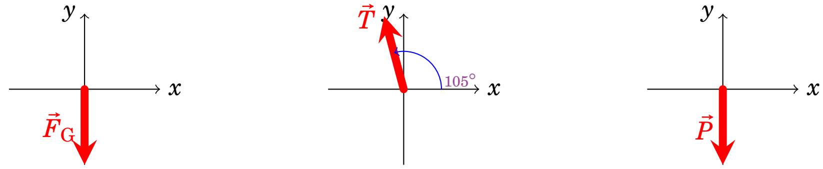
Step 0: The object is the person's whole arm.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull vertically downwards on the arm. The person is pulling upwards on the weight (the action), and so the weight pulls downwards on their fist (the reaction). The deltiods pull upwards to balance the torque due to the force at the fist. This presses the bones into contact in the shoulder joint.

Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for a complex system like this portion of the human body can be draw schematically (as a cartoon):



Step 3: The components of the forces acting are



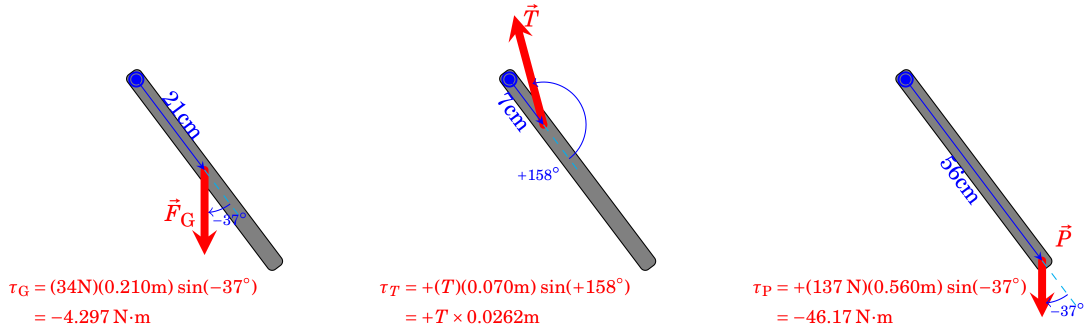
$F_{\text{G},x} = 0\,\text{N}$
$F_{\text{G},y} = -34\,\text{N}$

$T_x = T\cos 105°$
$T_y = T\sin 105°$

$P_x = 0\,\text{N}$
$P_y = -137\,\text{N}$

The components of the force at the joint remain as unknowns that we must solve for.

The contributions to torque from the forces acting on the object are:



As is usual, there is no contribution to torque from the force exerted at the joint.

Step 4: Since the arm is in static equilibrium the sum of forces must be zero:

$$\vec{F}_G + \vec{T} + \vec{P} + \vec{F}_J = \vec{0}\,\text{N} \tag{2.75}$$

The x-component of the sum is

$$F_{G,x} + T_x + P_x + F_{J,x} = 0\,\text{N} \tag{2.76}$$
$$0\,\text{N} + T\cos 105° + 0\,\text{N} + F_{J,x} = 0\,\text{N} \tag{2.77}$$

The y-component of the sum is

$$F_{G,y} + T_y + P_y + F_{J,y} = 0\,\text{N} \tag{2.78}$$
$$-34\,\text{N} + T\sin 105° - 137\,\text{N} + F_{J,y} = 0\,\text{N} \tag{2.79}$$

The sum of torques must also be zero:

$$\tau_G + \tau_T + \tau_P + \tau_J = 0\,\text{N·m} \tag{2.80}$$
$$-4.297\,\text{N·m} + T \times 0.0262\text{m} - 46.17\,\text{N·m} + 0\,\text{N·m} = 0\,\text{N·m} \tag{2.81}$$

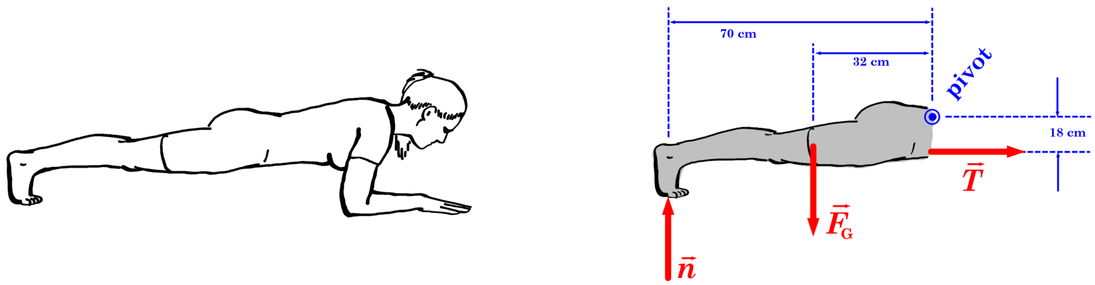
Step 5: Solving the sum-of-torques equation gives us $T = 1926\,\text{N}$. Using this in the sum-of-forces equations gives

$$F_{J,x} = +499\,\text{N} \tag{2.82}$$
$$F_{J,y} = -1689\,\text{N} \tag{2.83}$$

Answer: The magnitude of the force exerted by the muscles has to be $T = 1926\,\text{N}$. The force at the joint has magnitude $F_J = 1761\,\text{N}$, and points down and towards the right (about 16° to the right of the $-y$-axis).

PROBLEM 2.3.09:

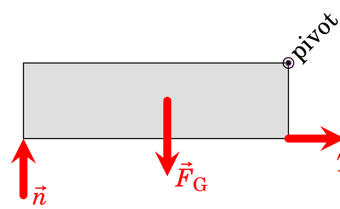
A woman doing core exercises is in “plank position”, as shown in the diagram below. The joint where the spine meets the pelvis is the pivot, with the legs on one side and the upper body on the other. It is the abdominal muscles that keep the two segments from bending away from each other. The contact force at her feet is 230.0 N, and the mass of the lower segment of her body is 30.61 kg. What is the tension in her abdominal muscles that keeps her body in static equilibrium?



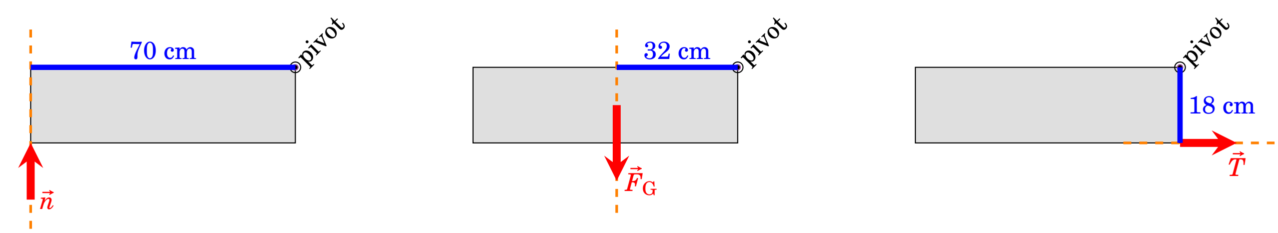
Step 0: The object is the lower portion of the person’s body.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull vertically downwards on this portion of their body. Their feet are touching the floor, so there is a normal force there. The abdominals pull towards the right to balance the torques due to the other forces. This presses the pelvis into contact with the lumbar vertebrae.

Step 2: Using a very crude cartoon of the object (depicting the lower half of their body by a rectangle), the Free-Body Diagram is:



Step 3: The components of the forces and the contributing torques are as follows:



$\tau_{\text{normal}} = -230.0\,\text{N} \times 0.70\,\text{m}$ $\quad$ $\tau_{\text{gravity}} = +300.0\,\text{N} \times 0.32\,\text{m}$ $\quad$ $\tau_{\text{abs}} = +T \times 0.18\,\text{m}$

Step 4: There is an unknown force ($\vec{F}_P$) acting at the pivot. The forces acting on this lower segment of the body must sum to zero:

$$n_x + F_{G,x} + T_x + F_{P,x} = 0\,\text{N} \qquad n_y + F_{G,y} + T_y + F_{P,y} = 0\,\text{N} \tag{2.84}$$

$$(0\,\text{N}) + (0\,\text{N}) + T_x + F_{P,x} = 0\,\text{N} \qquad (+230.0\,\text{N}) + (-300.0\,\text{N}) + (0\,\text{N}) + F_{P,y} = 0\,\text{N} \tag{2.85}$$

The sum of torques about the pivot is zero:

$$\tau_{\text{normal}} + \tau_{\text{gravity}} + \tau_{\text{abs}} = 0\,\text{N}\cdot\text{m} \tag{2.86}$$

$$(-230.0\,\text{N} \times 0.70\,\text{m}) + (+300.0\,\text{N} \times 0.32\,\text{m}) + (+T \times 0.18\,\text{m}) = 0\,\text{N}\cdot\text{m} \tag{2.87}$$

Step 5: In the sum of forces equations there are *three* unknowns: the tension and both components of the force at the pivot. Thus we must *first* solve for the tension. Using the sum of torques we can solve for the unknown tension in the abdominal muscles:

$$(-230.0\,\mathrm{N}\times 0.70\,\mathrm{m})+(+300.0\,\mathrm{N}\times 0.32\,\mathrm{m})+(+T\times 0.18\,\mathrm{m})=0\,\mathrm{N\cdot m} \tag{2.88}$$

$$T=361.1\,\mathrm{N} \tag{2.89}$$

With that value in the sum of forces equations, we can now solve for the unknown force acting at the pivot:

$$(0\,\mathrm{N})+(0\,\mathrm{N})+(+361.1\,\mathrm{N})+F_{\mathrm{P},x}=0\,\mathrm{N} \qquad (+230.0\,\mathrm{N})+(-300.0\,\mathrm{N})+(0\,\mathrm{N})+F_{\mathrm{P},y}=0\,\mathrm{N} \tag{2.90}$$

$$F_{\mathrm{P},x}=-361.1\,\mathrm{N} \qquad F_{\mathrm{P},y}=+70.0\,\mathrm{N} \tag{2.91}$$

The magnitude of this force is $F_\mathrm{P}=\sqrt{(-361.1\,\mathrm{N})^2+(+70.0\,\mathrm{N})^2}=367.8\,\mathrm{N}$.

From the signs of the components of this force, we see that it points into the second quadrant ($90^\circ<\theta<180^\circ$). Using the inverse tangent, our calculator gives us

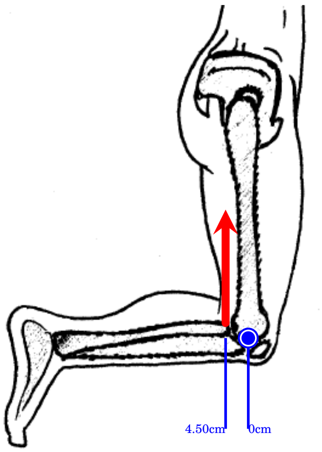
$$\theta=\tan^{-1}\left(\frac{F_{\mathrm{P},y}}{F_{\mathrm{P},x}}\right)=\tan^{-1}\left(\frac{+70.0\,\mathrm{N}}{-361.1\,\mathrm{N}}\right)=-11^\circ \tag{2.92}$$

which points into the fourth quadrant. The correct angle is $\theta=180^\circ-11^\circ=169^\circ$.

Answer: The tension in her abdominal muscles is 361 N. The force at the joint points towards her feet (to the left) and slightly upwards (169°) with a magnitude of 368 N.

PROBLEM 2.3.10:

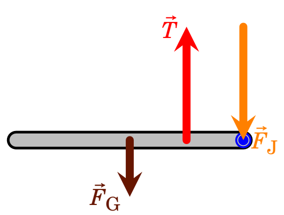
A person is standing vertically. Using their *hamstrings* (three muscles, of which the largest is the *biceps femoris*) they are holding the lower portion of one of their legs (3.00kg) horizontal, as shown. The muscles are attached 4.50cm from the pivot (the knee joint), and the center of mass of this segment is 15.3cm from the joint. What is the tension in the *hamstrings*?

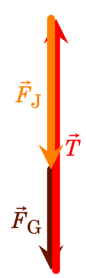


Step 0: The object is the person's lower leg, including the foot.

Step 1: The object is interacting with the following things: The Earth exerts a gravitational pull vertically downwards on the leg. Their hamstrings pull upwards to balance the torque due to gravity. This presses the bones into contact in the knee joint.

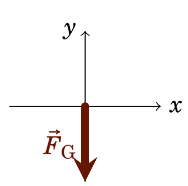
Step 2: For the purposes of finding the sum of forces and torques, the Free-Body Diagram for this portion of the human body can be draw schematically (as a cartoon):



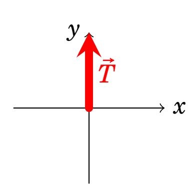


Since the tension in the muscle is about three times closer to the joint than gravity, for the torques about the joint to sum to zero, the tension in the muscle must be that much larger. Because of that, the force at the joint must be acting *downwards* for the sum of forces to be zero.

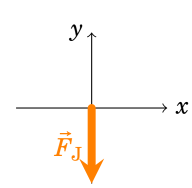
Step 3: Two of the three forces are unknown, but we do know that all three forces are vertical.



$F_{\text{G},x} = 0\,\text{N}$
$F_{\text{G},y} = -(3.00\,\text{kg})(9.81\,\frac{\text{m}}{\text{s}^2}) = -29.43\,\text{N}$



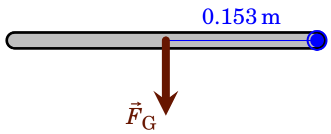
$T_x = 0\,\text{N}$
$T_y = +T$



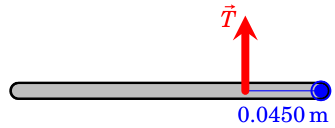
$F_{\text{J},x} = 0\,\text{N}$
$F_{\text{J},y} = -J$

The magnitude of the tension T in the hamstring, and the magnitude J of the force acting at the joint, are the unknowns we need to solve for.

The contributions to torque from the forces acting on the object are:



$$\tau_{G,z} = +(29.43\,\mathrm{N})(0.153\,\mathrm{m}) = +4.503\,\mathrm{N{\cdot}m}$$



$$\tau_{T,z} = -T \times 0.0450\,\mathrm{m}$$

Since it acts at the pivot the force at the joint contributes no torque.

Step 4: The forces acting on this lower segment of the body must sum to zero. Since all forces are vertical we need only consider the y-components:

$$F_{G,y} + T_y + F_{J,y} = 0\,\mathrm{N} \tag{2.93}$$
$$(-29.43\,\mathrm{N}) + (+T) + (-J) = 0\,\mathrm{N} \tag{2.94}$$

The sum of torques about the pivot is zero:

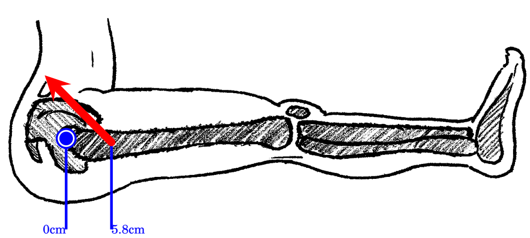
$$\tau_{G,z} + \tau_{T,z} + \tau_{J,z} = 0\,\mathrm{N{\cdot}m} \tag{2.95}$$
$$(+4.503\,\mathrm{N{\cdot}m}) + (-T \times 0.0450\,\mathrm{m}) + (0\,\mathrm{N{\cdot}m}) = 0\,\mathrm{N{\cdot}m} \tag{2.96}$$

Step 5: Solving the torque equation for the tension gets us $T = 100\,\mathrm{N}$. Substituting that into the sum of forces, we solve for the force at the joint and find $J = 70.6\,\mathrm{N}$.

Answer: The tension in their hamstrings is 100 N, and the force at the joint is 70.6 N pointing downwards.

PROBLEM 2.3.11:

A vertical person is holding one of their legs horizontally, as shown. One of the major flexors, the *iliopsoas*, connects the *lesser trochanter* (a small bump on the femur) to portions of the pelvis and the lumbar region of the spine. This muscle exerts a net force pointed at 57° above the horizontal (as shown), acting 5.8 cm from the hip joint. The whole leg has a mass of 10.40 kg, and the center of mass is 29.2 cm from the hip joint. What is the tension in the muscle?



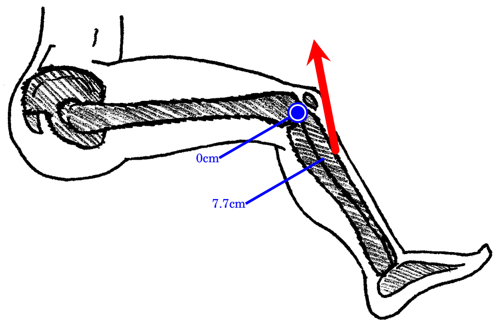
Step 0:

Step 1: Step 2: Step 3: Step 4: Step 5:

Answer: The tension in their *iliopsoas* is 612 N.

PROBLEM 2.3.12:

A vertical person with one of their legs as shown, is holding their lower leg at an angle of 30° from the vertical. This segment has a mass of 2.74 kg, with its center of mass 14.8 cm from the knee joint. The patellar ligament (the ligament that connects the patella [kneecap] to the tibia [shinbone]) attaches to the tibia at a point 7.7 cm from the joint, and the angle between the ligament and the tibia is 19°. What is the tension in the ligament?



Step 0: Step 1: Step 2: Step 3: Step 4: Step 5:

Answer: The tension in their patellar ligament is 79.3 N.

Chapter 3

Materials

Remember that the symbol σ means "stress" (the way in which force is distributed across the cross-section of the object $\sigma = F/A$) and that the symbol ϵ means "strain" (the relative change in length due to the applied stress $\epsilon = \Delta L/L_{\mathrm{i}}$). Unless otherwise explicitly stated all the stresses are axial tension or axial compression.

In exercises involving circular geometry remember to find the *radius* in cases when the diameter is specified. When the cross-section is hollow, obtain the area for the stress by *subtracting* the area of the hollow from the area of the cross-section.

Since strain is the change in length divided by the original length ($\epsilon = \Delta L/L = (L_{\mathrm{f}} - L_{\mathrm{i}})/L_{\mathrm{i}}$), the new length of the deformed object is

$$L_{\mathrm{f}} = L_{\mathrm{i}} + \Delta L \tag{3.1}$$
$$L_{\mathrm{f}} = L_{\mathrm{i}} + \epsilon L_{\mathrm{i}} \tag{3.2}$$

Note also that since strain is a ratio of lengths, the units used do not matter, as long as they are the same.

In your calculations: be very careful with the units and powers of ten; keep at least four significant figures in your intermediate steps; and round only the final result.

3.1 Stress

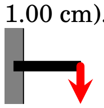
Exr 3.1.01 A 5.0 N force is applied across a rectangular surface measuring 2.0 cm by 7.0 cm. The stress is σ = 3.6 kPa.
The cross-sectional area is $A = (0.020\,\mathrm{m}) \times (0.070\,\mathrm{m}) = 1.4 \times 10^{-3}\,\mathrm{m}^2$.
The stress is thus $\sigma = F/A = (5.0\,\mathrm{N})/(1.4 \times 10^{-3}\,\mathrm{m}^2) = 3.571 \times 10^{+3}\,\mathrm{Pa} = 3.6\,\mathrm{kPa}$, written to two significant figures.

Exr 3.1.02 A 1060 N force is applied across a rectangular surface measuring 10 cm by 30 cm. The stress is σ = 35 kPa.
The cross-sectional area is $A = (0.10\,\mathrm{m}) \times (0.30\,\mathrm{m}) = 3.0 \times 10^{-2}\,\mathrm{m}^2$.
The stress is thus $\sigma = F/A = (1060\,\mathrm{N})/(3.0 \times 10^{-2}\,\mathrm{m}^2) = 3.533 \times 10^{+4}\,\mathrm{Pa} = 35\,\mathrm{kPa}$, written to two significant figures.

Exr 3.1.03 A 120 kg mass is resting on a rectangular surface measuring 8.0 cm by 27.0 cm. The pressure is P = 55 kPa.
Recall that pressure on a surface is just another name for compressive stress. The force applied is the weight: $F = mg = (120\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 1177.2\,\mathrm{N}$. (Remember: do not round your intermediate calculations!) The cross-sectional area is $A = (0.080\,\mathrm{m}) \times (0.270\,\mathrm{m}) = 2.16 \times 10^{-2}\,\mathrm{m}^2$. The stress is thus $\sigma = F/A = (1177.2\,\mathrm{N})/(2.16 \times 10^{-2}\,\mathrm{m}^2) = 5.450 \times 10^{+4}\,\mathrm{Pa} = 55\,\mathrm{kPa}$, written to two significant figures.

Exr 3.1.04 A force of 33.7 N is shearing across a rectangular surface measuring 5.0 mm by 8.0 mm. The shear stress is σ = 0.84 MPa.
The cross-sectional area is $A = (0.0050\,\mathrm{m}) \times (0.0080\,\mathrm{m}) = 4.0 \times 10^{-5}\,\mathrm{m}^2$.
The stress is thus $\sigma = F/A = (33.7\,\mathrm{N})/(4.0 \times 10^{-5}\,\mathrm{m}^2) = 8.425 \times 10^{+5}\,\mathrm{Pa} = 0.84\,\mathrm{MPa}$, written to two significant figures.

Exr 3.1.05 A 7.16 kg mass is suspended at the end of a horizontal metal rod that is square in cross-section (width 1.00 cm). The *shear* stress is σ = 0.702 MPa.

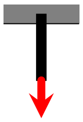


The force applied is the weight: $F = mg = (7.16\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 70.24\,\mathrm{N}$. The cross-sectional area is $A = (0.0100\,\mathrm{m})^2 = 1.00 \times 10^{-4}\,\mathrm{m}^2$. The stress is thus $\sigma = F/A = (70.24\,\mathrm{N})/(1.00 \times 10^{-4}\,\mathrm{m}^2) = 7.024 \times 10^{+5}\,\mathrm{Pa} = 0.702\,\mathrm{MPa}$, written to three significant figures.

Exr 3.1.06 A cylindrical rod of diameter 5.00 cm is under 12.0 N of compression. The stress is σ = 6.11 kPa.
Since the diameter of the rod is 5.00 cm, its radius is $r = 2.50\,\mathrm{cm} = 0.0250\,\mathrm{m}$, and its cross-sectional area is $\pi r^2 =$

$\pi(0.0250\,\text{m})^2 = 1.9635\times10^{-3}\,\text{m}^2$. The stress is thus $\sigma = F/A = (12.0\,\text{N})/(1.9635\times10^{-3}\,\text{m}^2) = 6.11\,\text{kPa}$, written to three significant figures.

EXR 3.1.07 A 621 kg mass hangs at the end of a 12.7 mm diameter metal rod. The stress is σ = 48.1 MPa.



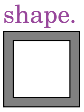
Since the diameter of the rod is 12.7 mm, its radius is $r = 6.35\,\text{mm} = 6.35\times10^{-3}\,\text{m}$, and its cross-sectional area is $\pi r^2 = \pi(6.35\times10^{-3}\,\text{m})^2 = 1.2668\times10^{-4}\,\text{m}^2$. The force applied is the weight: $F = mg = (621\,\text{kg})(9.81\,\text{N/kg}) = 6092\,\text{N}$. The stress is thus $\sigma = F/A = (6092\,\text{N})/(1.2668\times10^{-4}\,\text{m}^2) = 48.1\,\text{MPa}$, written to three significant figures.

EXR 3.1.08 A 7.00 kg mass hangs at the end of a cylindrical rod of diameter 6.0 mm. The stress is σ = 2.4 MPa.
Since the diameter of the rod is 6.0 mm, its radius is $r = 3.0\,\text{mm} = 3.0\times10^{-3}\,\text{m}$, and its cross-sectional area is $\pi r^2 = \pi(3.0\times10^{-3}\,\text{m})^2 = 2.827\times10^{-5}\,\text{m}^2$. The force applied is the weight: $F = mg = (7.00\,\text{kg})(9.81\,\text{N/kg}) = 68.67\,\text{N}$. The stress is thus $\sigma = F/A = (68.67\,\text{N})/(2.827\times10^{-5}\,\text{m}^2) = 2.4\,\text{MPa}$, written to two significant figures.

EXR 3.1.09 A 7.16 kg mass is suspended at the end of a horizontal metal rod that is circular in cross-section (diameter 1.00 cm). The *shear* stress is σ = 0.89 MPa.
Since the diameter of the rod is 1.00 cm, its radius is $r = 0.50\,\text{cm} = 5.0\times10^{-3}\,\text{m}$, and its cross-sectional area is $\pi r^2 = \pi(5.0\times10^{-3}\,\text{m})^2 = 7.854\times10^{-5}\,\text{m}^2$. The force applied is the weight: $F = mg = (7.16\,\text{kg})(9.81\,\text{N/kg}) = 70.24\,\text{N}$. The stress is thus $\sigma = F/A = (70.24\,\text{N})/(7.854\times10^{-5}\,\text{m}^2) = 0.89\,\text{MPa}$, written to two significant figures.

EXERCISE 3.1.10 A hollow square beam supports 12 507 N of weight. The outer width of the beam is 42.7 mm and the inner width of the hollow is 40.1 mm. The stress in the material of the beam is σ = 58.1 MPa.

The area of *material* in the cross-section of a hollow object is the area of the outer shape minus the area of the inner shape. This is the picture of the cross-section:

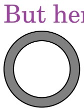


In this exercise the outer area is $A_{\text{outer}} = (0.0427\,\text{m})^2 = 1.82329\times10^{-3}\,\text{m}^2$, and the inner area is $A_{\text{inner}} = (0.0401\,\text{m})^2 = 1.60801\times10^{-3}\,\text{m}^2$. So the area of the material in the cross-section is $A = A_{\text{outer}} - A_{\text{inner}} = 1.82329\times10^{-3}\,\text{m}^2 - 1.60801\times10^{-3}\,\text{m}^2 = 2.1528\times10^{-4}\,\text{m}^2$. (Notice that we should keep extra decimal places in the intermediate steps of our calculations when taking a difference.)

The stress in the material is thus $\sigma = F/A = (12507\,\text{N})/(2.1528\times10^{-4}\,\text{m}^2) = 58.1\,\text{MPa}$ written to three significant figures.

EXERCISE 3.1.11 A 18.755 kg bowling ball rests on the end of a hollow cylinder with outer radius 3.30 cm and inner radius 3.20 cm. The stress in the material of the cylinder is σ = 0.901 MPa.

Usually, when working with a circular cross-section, we need to be careful to obtain the radius from the diameter. But here we are given the radius (for both the inner and outer areas). This is the picture of the cross-section:

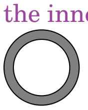


The area of material in the cross-section is the area of outer minus the inner. The outer area is $A_{\text{outer}} = \pi(0.0330\,\text{m})^2 = 3.421\times10^{-3}\,\text{m}^2$, and the inner area is $A_{\text{inner}} = \pi(0.0320\,\text{m})^2 = 3.217\times10^{-3}\,\text{m}^2$. So the area of the material in the cross-section is $A = A_{\text{outer}} - A_{\text{inner}} = 3.421\times10^{-3}\,\text{m}^2 - 3.217\times10^{-3}\,\text{m}^2 = 0.204\times10^{-3}\,\text{m}^2$.

The force applied is the weight: $F = mg = (18.755\,\text{kg})(9.81\,\text{N/kg}) = 184\,\text{N}$. The stress in the material is thus $\sigma = F/A = (184\,\text{N})/(0.204\times10^{-3}\,\text{m}^2) = 9.01\times10^{+5}\,\text{Pa} = 0.901\times10^{+6}\,\text{Pa} = 0.901\,\text{MPa}$ written to three significant figures.

EXERCISE 3.1.12 A hollow metal tube with outer diameter 3.81 mm and inner diameter 3.77 mm is under a tension of 522 N. The stress in the material of the tube is σ = 2.19 GPa.

We are given two diameters; the outer and inner measurements of the tube. Given these diameters, the radii are $r_{\text{outer}} = 1.905\times10^{-3}\,\text{m}$ and $r_{\text{inner}} = 1.885\times10^{-3}\,\text{m}$. The area of material in the cross-section is the area of outer minus the inner. This is the picture of the cross-section:



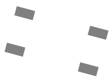
The outer area is $A_{\text{outer}} = \pi(1.905\times10^{-3}\,\text{m})^2 = 11.40092\times10^{-6}\,\text{m}^2$, and the inner area is $A_{\text{inner}} = \pi(1.885\times10^{-3}\,\text{m})^2 =$

$11.16279 \times 10^{-6}\,\mathrm{m}^2$. (Here, because the radii are so close to each other in value we need to keep extra decimal places when we take the difference, and only round *after*.) So the area of the material in the cross-section is $A = A_{\text{outer}} - A_{\text{inner}} = 11.40092 \times 10^{-6}\,\mathrm{m}^2 - 11.16279 \times 10^{-6}\,\mathrm{m}^2 = 0.23813 \times 10^{-6}\,\mathrm{m}^2$.

The stress in the material is thus $\sigma = F/A = (522\,\mathrm{N})/(0.23813 \times 10^{-6}\,\mathrm{m}^2) = 2.19 \times 10^{+9}\,\mathrm{Pa} = 2.19\,\mathrm{GPa}$ written to three significant figures.

EXERCISE 3.1.13 A 1406 kg car rests on its four tires, each with a contact patch measuring 19 cm by 24 cm. The pressure on each tire is P = 76 kPa.

The weight of the car is spread over an area that has four separate segments: the areas of contact where the four wheels contact the road. This is the picture of the areas over which the net force is distributed:



Each separate contact patch has an area $A = 0.19\,\mathrm{m} \times 0.24\,\mathrm{m} = 0.0456\,\mathrm{m}^2$. The total area is thus $A = 4 \times 0.0456\,\mathrm{m}^2 = 0.182\,\mathrm{m}^2$. The weight is $F = (1406\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 13\,792\,\mathrm{N}$. The pressure of contact is thus $P = F/A = (13\,792\,\mathrm{N})/(0.182\,\mathrm{m}^2) = 7.562 \times 10^{+4}\,\mathrm{Pa} = 76\,\mathrm{kPa}$ written to two significant figures.

The question asks about "the pressure on each tire". We do *not* divide this result by four. This pressure is the result of spreading the weight across all four of the tires, and so each tire has this same value of pressure.

EXR 3.1.14 A 54.4 kg ballerina stands on one foot (equivalent to a rectangle measuring 20.3 cm by 5.1 cm). The pressure under her foot is P = 52 kPa.
This is not a question of stress *inside* a material. This is about how a force of contact is distributed across a surface. This is not a stress, it is a pressure: $P = F/A$. The force is her weight $(54.4\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 533.66\,\mathrm{N}$, and the area is $0.203\,\mathrm{m} \times 0.051\,\mathrm{m} = 0.010\,353\,\mathrm{m}^2$. The pressure under her foot is thus $P = F/A = (533.66\,\mathrm{N})/(0.010\,353\,\mathrm{m}^2) = 5.1547 \times 10^{+4}\,\mathrm{Pa} = 52\,\mathrm{kPa}$.

EXR 3.1.15 A 54.4 kg ballerina stands on the point of her toes of one foot (equivalent to a rectangle measuring 6.0 cm by 2.0 cm). The pressure under her toes is P = 0.44 MPa.
The force is her weight $(54.4\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 533.66\,\mathrm{N}$, and the area is $0.060\,\mathrm{m} \times 0.020\,\mathrm{m} = 0.0012\,\mathrm{m}^2$. The pressure under her foot is thus $P = F/A = (533.66\,\mathrm{N})/(0.0012\,\mathrm{m}^2) = 4.4472 \times 10^{+5}\,\mathrm{Pa} = 0.44\,\mathrm{MPa}$.

EXR 3.1.16 An adult male moose can have masses up to 700 kg! Their femur is a hollow cylinder of bone, outer diameter 5.5 cm, inner diameter 3.5 cm. The stress in the bone is σ = 1.2 MPa. (Assume that its weight is distributed equally onto each of their four legs.)

This is the picture of the cross-sectional area of one femur over which *one quarter* of the animal's weight is distributed:

The cross-sectional area of this femur is $A_{\text{one}} = A_{\text{outer}} - A_{\text{inner}} = \pi(\frac{1}{2} \times 0.055\,\mathrm{m})^2 - \pi(\frac{1}{2} \times 0.035\,\mathrm{m})^2 = 1.4137 \times 10^{-3}\,\mathrm{m}^2$.

Their weight is $(700\,\mathrm{kg})(9.81\,\mathrm{N/kg}) = 6\,867\,\mathrm{N}$, and this femur carries a quarter of that: $F = \frac{1}{4} \times 6\,867\,\mathrm{N} = 1\,717\,\mathrm{N}$. So the stress in the material of each bone is $\sigma = F/A = (1\,717\,\mathrm{N})/(1.4137 \times 10^{-3}\,\mathrm{m}^2) = 1.21\,\mathrm{MPa}$ written to three significant figures.

3.2 Strain

While doing these exercises and problems remember:

- Strain is defined to be a ratio of lengths, so make sure that both quantities have the same units.
- Do not round any of the values you are using until you reach the final result. The change in length will be the subtraction of two numbers that (typically) differ only in their last few decimal places; do not truncate any of those.
- The sign of the change in length, and hence the sign of the strain, is *very* important. Double-check that it is correct: $\epsilon > 0$ means an increase in length, while $\epsilon < 0$ means a decrease in length. In your answer be explicit about the sign.
- With the exception of extremely soft elastic materials (like rubber) realistic strains should be very small numbers. If you find a strain $\epsilon > 1$, then you've probably made a mistake somewhere.

EXR 3.2.01 A meter stick (length 1.000 m) is subjected to tension and stretches to 1.003 m. The strain is $\epsilon = +3 \times 10^{-3}$.

The amount of length change is $\Delta L = L_f - L_i = 1.003\,\text{m} - 1.000\,\text{m} = +0.003\,\text{m}$. Notice how there is only one significant figure here! This means that the strain is $\epsilon = \frac{\Delta L}{L_i} = \frac{+0.003\,\text{m}}{1.000\,\text{m}} = +0.003 = +3 \times 10^{-3}$, to one significant figure.

Exr 3.2.02 An elastic band is stretched from 8.5 cm to 14.2 cm. The strain is ϵ = +0.67.
The amount of length change is $\Delta L = L_f - L_i = 14.2\text{ cm} - 8.5\text{ cm} = +5.7\text{ cm}$. This means that the strain is $\epsilon = \frac{\Delta L}{L_i} = \frac{+5.7\,\text{cm}}{8.5\,\text{cm}} = +0.670588\ldots = +0.67$, to two significant figures. Notice how large this is in comparison to the values found in the other exercises that involve solid objects. Large strains like this are only possible in materials like rubbers and other polymers.

Exr 3.2.03 An object of length 45.00 cm is subjected to a stress and compresses to 44.92 cm. The strain is $\epsilon = -2 \times 10^{-3}$.
The amount of length change is $\Delta L = L_f - L_i = 44.92\,\text{cm} - 45.00\,\text{cm} = -0.08\,\text{cm}$. Notice how there is only one significant figure here. This means that the strain is $\epsilon = \frac{\Delta L}{L_i} = \frac{-0.08\,\text{cm}}{45.00\,\text{cm}} = -0.001777\ldots = -0.002 = -2 \times 10^{-3}$, to one significant figure.

Exr 3.2.04 A steel ruler (length 12.02 in) is subjected to tension and stretches to 12.03 in. The strain is ϵ = +0.08%. (We are looking for the strain expressed as a percent.)
The amount of length change is $\Delta L = L_f - L_i = 12.03\,\text{in} - 12.02\,\text{in} = +0.01\,\text{in}$. Notice how there is only one significant figure here. This means that the strain is $\epsilon = \frac{\Delta L}{L_i} = \frac{+0.01\,\text{in}}{12.02\,\text{in}} = +0.0008319\ldots = +0.0008 = +\frac{0.08}{100} = +0.08\%$, to one significant figure.

(The lengths in this exercise were are given in the imperial unit of inches. There was no need to convert these to metres since they cancel when we take the ratio to calculate the strain. As long as the units of ΔL and L_i are the same they will cancel when calculating ϵ.)

Exr 3.2.05 A wooden column of height 67.02 inches is subjected to a stress and compresses to 66.97 inches. The strain is $\epsilon = -7 \times 10^{-4}$.
The amount of length change is $\Delta L = L_f - L_i = 66.97\,\text{in} - 67.02\,\text{in} = -0.05\,\text{in}$. Notice how there is only one significant figure here. This means that the strain is $\epsilon = \frac{\Delta L}{L_i} = \frac{-0.05\,\text{in}}{67.02\,\text{in}} = -0.0007460\ldots = -7 \times 10^{-4}$, to one significant figure.

Exr 3.2.06 A ruler (original length 30.22 cm) is subjected to tension and exhibits a strain of ϵ = +0.20%. The deformed length is L_f = 30.28 cm.
The strain, expressed as number, rather than a percentage, is $\epsilon = +0.20\% = +\frac{0.20}{100} = +0.0020$. The change in length is $\Delta L = \epsilon L_i = (+0.0020)(30.22\,\text{cm}) = +0.06044\,\text{cm}$. The final length is thus $L_f = L_i + \Delta L = (30.22\,\text{cm}) + (+0.06044\,\text{cm}) = 30.28\,\text{cm}$, to two decimal places.

Exr 3.2.07 A segment of a bridge (original length 27.052 m) is subjected to a stress and exhibits a strain of $\epsilon = +3.5 \times 10^{-5}$. The deformed length is L_f = 27.053 m.
The change in length is $\Delta L = \epsilon L_i = (+3.5 \times 10^{-5})(27.052\,\text{m}) = +9.4682 \times 10^{-6}\,\text{m}$. The final length is thus $L_f = L_i + \Delta L = (27.052\,\text{m}) + (+9.4682 \times 10^{-6}\,\text{m}) = 27.053\,\text{m}$, to three decimal places.

Exr 3.2.08 A segment of a bridge (original length 27.052 m) is subjected to a stress and exhibits a strain of $\epsilon = -3.5 \times 10^{-5}$. The deformed length is L_f = 27.051 m.
The change in length is $\Delta L = \epsilon L_i = (-3.5 \times 10^{-5})(27.052\,\text{m}) = -9.4682 \times 10^{-6}\,\text{m}$. The final length is thus $L_f = L_i + \Delta L = (27.052\,\text{m}) + (-9.4682 \times 10^{-6}\,\text{m}) = 27.051\,\text{m}$, to three decimal places.

Exr 3.2.09 When the ground under a building subsides a concrete wall (original height 13.09 m) is subjected to a stress and exhibits a strain of $\epsilon = +7.30 \times 10^{-6}$. The deformed height is L_f = 13.09 m.
The change in height is $\Delta L = \epsilon L_i = (+7.30 \times 10^{-6})(13.09\text{ m}) = +95.6 \times 10^{-6}\text{ m}$. The final height is thus $L_f = L_i + \Delta L = (13.09\text{ m}) + (+95.6 \times 10^{-6}\text{ m}) = 13.0900956\text{ m} = 13.09\text{ m}$, to two decimal places. The strain is not zero, and is measurable with sophisticated equipment, but is not significant on the human scale.

Exr 3.2.10 A person out on a nature hike takes a rest on a granite boulder (original height 75.00 cm). The compressive stress due to the person's weight causes a strain of $\epsilon = -5.28 \times 10^{-8}$ in the boulder. The deformed height is L_f = 75.00 cm. As with the previous exercise, the deformation is utterly negligible in comparison to the original size.

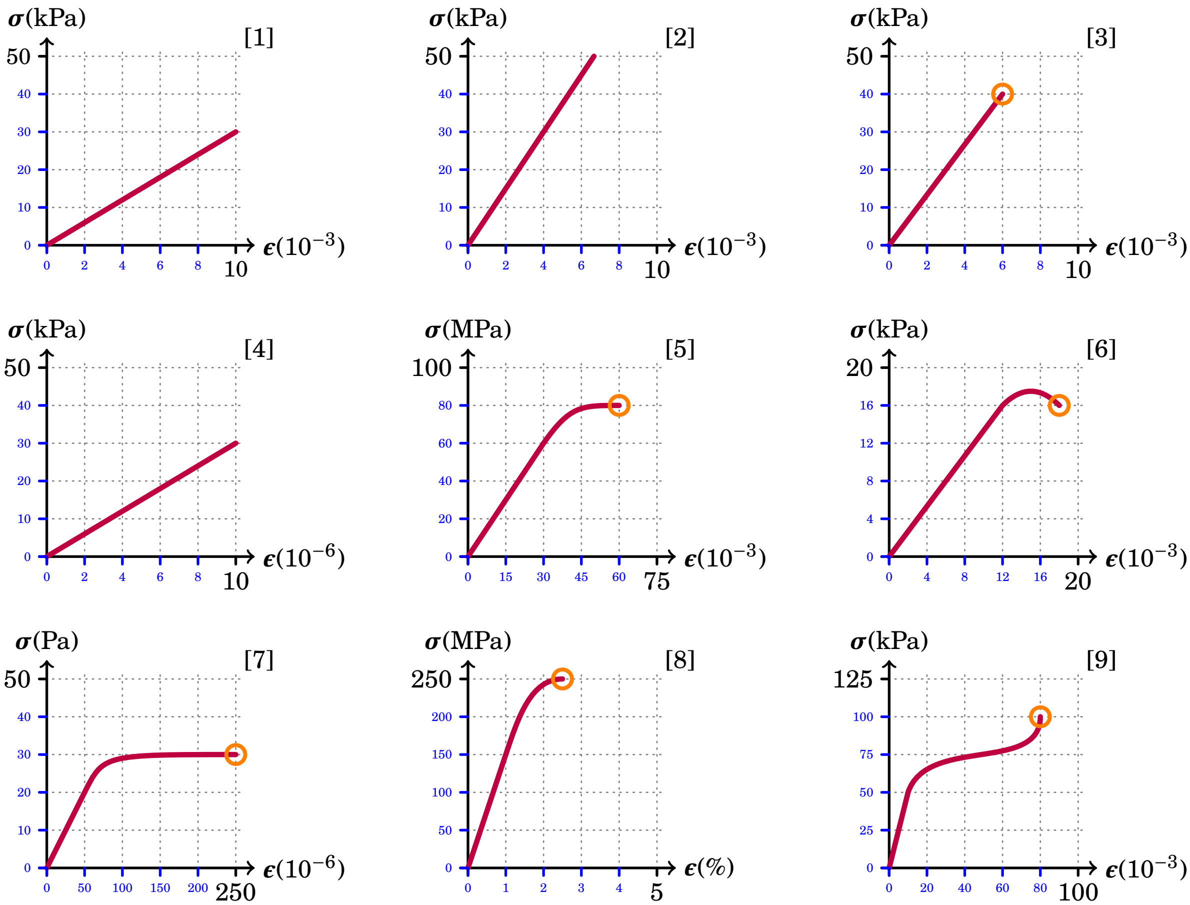
3.3 Stress-Strain Curves

In each of the stress-strain graphs of the following exercises a small orange circle around the end of a stress-strain curve indicates the failure of the material.

EXERCISE 3.3.01 Consider the stress-strain curves plotted below.

(a) Calculate the Young's Modulus for each of the materials. (Be careful with the units!)

(b) For those materials that have a failure in the plotted range of strains, estimate the stress and strain at failure. For those graphs also estimate their values at the yield point (if it exists).



General comments:

Part (a): Since the proportional regime of each of the stress-strain curves here begins at the origin, the Young's Modulus for each can be found by picking one point on its curve and finding the ratio: $\mathscr{Y} = \Delta\sigma/\Delta\epsilon = (\sigma - 0\text{Pa})/(\epsilon - 0) = \sigma/\epsilon$.

Part (b): Only the curves [1], [2] and [4] do not have the failure point of their materials plotted. Those curves also do not show their yield point.

[1] (a) Picking the point $\sigma = 30\,\text{kPa}$ and $\epsilon = 10 \times 10^{-3}$, we get
$\mathscr{Y} = (30 \times 10^{+3}\,\text{Pa})/(10 \times 10^{-3}) = 3.0 \times 10^{+6}\,\text{Pa} = 3.0\,\text{MPa}$.
(b) (No answer.)

[2] (a) Picking the point $\sigma = 30\,\text{kPa}$ and $\epsilon = 4 \times 10^{-3}$, we get
$\mathscr{Y} = (30 \times 10^{+3}\,\text{Pa})/(4 \times 10^{-3}) = 7.5 \times 10^{+6}\,\text{Pa} = 7.5\,\text{MPa}$.
(b) (No answer.)

[3] (a) Picking the point $\sigma = 40\,\text{kPa}$ and $\epsilon = 6 \times 10^{-3}$, we get
$\mathscr{Y} = (40 \times 10^{+3}\,\text{Pa})/(6 \times 10^{-3}) = 6.7 \times 10^{+6}\,\text{Pa} = 6.7\,\text{MPa}$.

(b) Failure happens at $\sigma = 40\,\text{kPa}$ and $\epsilon = 6 \times 10^{-3}$.
Of the given curves only [3] is brittle (it has no regime of plastic deformation before failure).

[4] **(a)** Picking the point $\sigma = 30\,\text{kPa}$ and $\epsilon = 10 \times 10^{-6}$, we get
$\mathscr{Y} = (30 \times 10^{+3}\,\text{Pa})/(10 \times 10^{-6}) = 3.0 \times 10^{+9}\,\text{Pa} = 3.0\,\text{GPa}$.
(b) (No answer.)

[5] **(a)** Picking the point $\sigma = 60\,\text{MPa}$ and $\epsilon = 30 \times 10^{-3}$, we get
$\mathscr{Y} = (60 \times 10^{+6}\,\text{Pa})/(30 \times 10^{-3}) = 2.0 \times 10^{+9}\,\text{Pa} = 2.0\,\text{GPa}$.
(b) Failure happens at $\sigma = 80\,\text{MPa}$ and $\epsilon = 60 \times 10^{-3}$, and
the yield point looks like it might be near $\sigma = 60\,\text{MPa}$ and $\epsilon = 30 \times 10^{-3} = 0.030$.

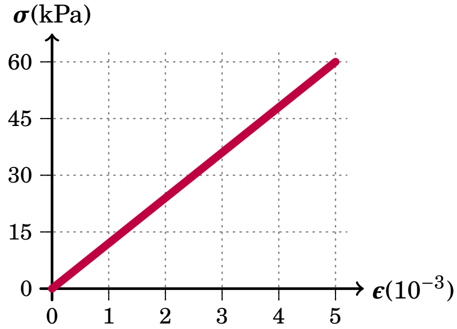
[6] **(a)** Picking the point $\sigma = 16\,\text{kPa}$ and $\epsilon = 12 \times 10^{-3}$, we get
$\mathscr{Y} = (16 \times 10^{+3}\,\text{Pa})/(12 \times 10^{-3}) = 1.3 \times 10^{+6}\,\text{Pa} = 1.3\,\text{MPa}$.
(b) Failure happens near $\sigma = 16\,\text{kPa}$ and $\epsilon = 18 \times 10^{-3}$, and
the yield point looks like it might be near $\sigma = 16\,\text{kPa}$ and $\epsilon = 12 \times 10^{-3} = 0.012$.

[7] **(a)** Picking the point $\sigma = 20\,\text{Pa}$ and $\epsilon = 50 \times 10^{-6}$, we get
$\mathscr{Y} = (20\,\text{Pa})/(50 \times 10^{-6}) = 4.0 \times 10^{+5}\,\text{Pa} = 0.40\,\text{MPa}$.
(b) Failure happens at $\sigma = 30\,\text{Pa}$ and $\epsilon = 250 \times 10^{-6}$, and
the yield point looks like it might be near $\sigma = 20\,\text{Pa}$ and $\epsilon = 50 \times 10^{-6}$.

[8] **(a)** Picking the point $\sigma = 150\,\text{MPa}$ and $\epsilon = 1\% = 0.01$, we get
$\mathscr{Y} = (150 \times 10^{+6}\,\text{Pa})/(0.01) = 15. \times 10^{+9}\,\text{Pa} = 15.\,\text{GPa}$.
(b) Failure happens at $\sigma = 250\,\text{MPa}$ and $\epsilon = 2.5\% = 0.025$, and
the yield point looks like it might be near $\sigma = 150\,\text{MPa}$ and $\epsilon = 0.01$.

[9] **(a)** Picking the point $\sigma = 50\,\text{kPa}$ and $\epsilon = 10 \times 10^{-3}$, we get
$\mathscr{Y} = (50 \times 10^{+3}\,\text{Pa})/(10 \times 10^{-3}) = 5.0 \times 10^{+6}\,\text{Pa} = 5.0\,\text{MPa}$.
(b) Failure happens at $\sigma = 100\,\text{kPa}$ and $\epsilon = 80 \times 10^{-3} = 0.080$, and
the yield point looks like it might be near $\sigma = 50\,\text{kPa}$ and $\epsilon = 10 \times 10^{-3} = 0.010$.

Problem 3.3.01:



Given the stress-strain curve on the right:

(a) What is the Young's modulus of the material?

The Young's modulus of the material is the slope of the stress-strain graph: $\mathscr{Y} = \Delta\sigma/\Delta\epsilon = (60\times10^{+3}\,\text{Pa})/(5\times10^{-3}) = 12\times10^{+6}\,\text{Pa} = 12\,\text{MPa}$.

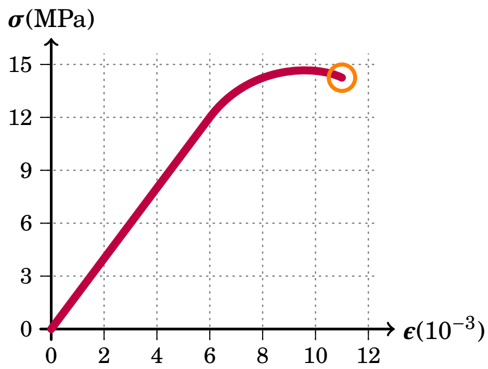
(b) What is stress in the material
when the strain is $\epsilon = 3.5\times10^{-3}$?

Visually, the line is somewhere near $\sigma \approx 40\,\text{kPa}$ when $\epsilon = 3.5\times10^{-3}$, but we can calculate it more precisely.
Since the curve begins at the origin $\Delta\epsilon = \epsilon - 0 = \epsilon$. From the definition of $\mathscr{Y}$ we can write $\Delta\sigma = \mathscr{Y}\times\Delta\epsilon = \mathscr{Y}\times\epsilon = (12\times10^{+6}\,\text{Pa})\times(3.5\times10^{-3}) = 42\times10^{+3}\,\text{Pa} = 42\,\text{kPa}$.

(c) What is strain in the material
when the stress is $\sigma = 20\,\text{kPa}$?

Visually, the line is somewhere between $1.5\times10^{-3} < \epsilon < 2.0\times10^{-3}$ when $\sigma = 20\,\text{kPa}$, but we can calculate it more precisely.
Since the curve begins at the origin $\Delta\sigma = \sigma - 0\,\text{Pa} = \sigma$. From the definition of $\mathscr{Y}$ we can write $\Delta\epsilon = \Delta\sigma/\mathscr{Y} = \sigma/\mathscr{Y} = (20\times10^{+3}\,\text{Pa})/(12\times10^{+6}\,\text{Pa}) = 1.7\times10^{-3}$, which is close to the estimate.

PROBLEM 3.3.02:



Given the stress-strain curve on the right:

(a) What is the strain at the yield point? If the object were 14.0 cm in length when unstressed, what would be its length at this limit?

Inspecting the graph, it looks to be at about $\epsilon = 6 \times 10^{-3}$. From the definition of strain $\Delta L = \epsilon \times L_{\mathrm{i}} = (6 \times 10^{-3}) \times (14.0 \times 10^{-2}\ \mathrm{m}) = 84 \times 10^{-5}\ \mathrm{m} = 0.084\ \mathrm{cm}$. So the length would be $L_{\mathrm{f}} = L_{\mathrm{i}} + \Delta L = 14.084\ \mathrm{cm}$.

(b) What is the Young's modulus of the material in the elastic regime?

The Young's modulus of the material is the slope of the Stress-Strain graph in the elastic regime (the straight-line portion near the origin): $\mathscr{Y} = \Delta\sigma/\Delta\epsilon = (12 \times 10^{+6}\ \mathrm{Pa})/(6 \times 10^{-3}) = 2 \times 10^{+9}\ \mathrm{Pa} = 2\ \mathrm{GPa}$.

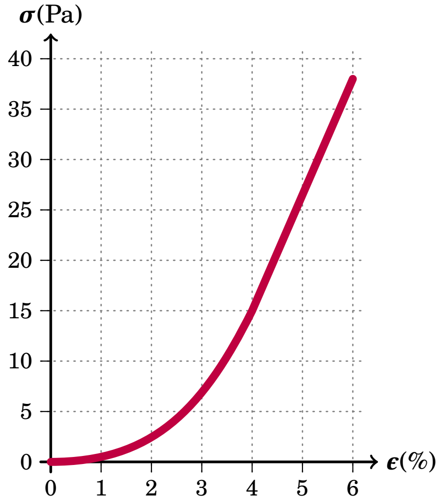
(c) What is stress in the material when the strain is $\epsilon = 3.0 \times 10^{-3}$?

Since the curve begins at the origin $\Delta\epsilon = \epsilon$. From the definition of $\mathscr{Y}$ we can write $\Delta\sigma = \mathscr{Y} \times \Delta\epsilon = \mathscr{Y} \times \epsilon = (2 \times 10^{+9}\ \mathrm{Pa}) \times (3.0 \times 10^{-3}) = 6 \times 10^{+6}\ \mathrm{Pa} = 6\ \mathrm{MPa}$.

(d) What is the strain at failure?

Inspecting the graph, it looks to be at about $\epsilon = 11 \times 10^{-3}$.

PROBLEM 3.3.03:



Given the stress-strain curve on the right:

(a) From the shape of the stress-strain curve, what *type* of material is this?

With the curved portion at small strain before the linear portion we recognize this as an elastomeric material (like tendons or ligaments).

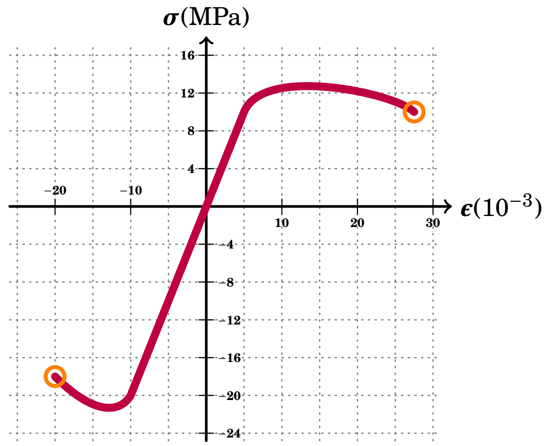
(b) Estimate the strain of the material when the stress is $\sigma = 10\,\text{Pa}$.

It looks like $\epsilon \approx 3.5\%$.

(c) Estimate the value of Young's modulus in the regime where the stress-strain curve is linear.

The linear portion of the graph looks like it is in the range $4\% < \epsilon < 6\%$. (Remember that "$\epsilon = 4\%$" means $\epsilon = 0.04$.) The Young's modulus in this regime is the slope of this segment of the graph: $\mathscr{Y} = \Delta\sigma / \Delta\epsilon \approx (38\,\text{Pa} - 15\,\text{Pa})/(0.06 - 0.04) = 1.15 \times 10^{+3}\,\text{Pa} = 1.2\,\text{kPa}$.

PROBLEM 3.3.04:



The stress-strain curve above shows the behavior of the material both under compression and under tension. Given that curve:

(a) Estimate the stress and strain at yield when the material is under tension.
$\epsilon \approx 5 \times 10^{-3}$ and $\sigma \approx 10\,\text{MPa}$.

(b) Estimate the stress and strain at yield when the material is under compression.
$\epsilon \approx -10 \times 10^{-3}$ and $\sigma \approx -20\,\text{MPa}$.

(c) Estimate the value of strain at which the stress has its largest magnitude. Is that when the material is under tension or compression?
Looking at the graph, the material sustains a larger magnitude of stress when it is compressed (achieving $\sigma \approx -21\,\text{MPa}$ under compression versus $\sigma \approx +13\,\text{MPa}$ under tension). This happens at $\epsilon \approx -13 \times 10^{-3}$, approximately.

(d) Estimate the stress and strain when the material fails under tension.
$\epsilon \approx 27 \times 10^{-3}$ and $\sigma \approx 10\,\text{MPa}$.

(e) Estimate the value of Young's modulus for values of strain between the compressive yield point and the tensile yield point.
The linear portion of the graph is between the two yield points, in the range $-10 \times 10^{-3} < \epsilon < +5 \times 10^{-3}$. The Young's modulus in this regime is the slope of this segment of the graph:

$$\mathscr{Y} = \Delta\sigma/\Delta\epsilon \approx \frac{(+10\,\text{MPa}) - (-20\,\text{MPa})}{(+5 \times 10^{-3}) - (-10 \times 10^{-3})} = 2.0 \times 10^{+9}\,\text{Pa} = 2.0\,\text{GPa}$$

Chapter 4

Energy

Introduction.

4.1 Thermal Energy

4.1.1 Units

Exr 4.1.01 What is the temperature 300. K measured in Celsius?
$T_C = T_K - 273.15\,°C = (300. - 273.15)°C = 26.85\,°C$

Exr 4.1.02 The SI system is based on the metre, kilogram, and second. In this system the unit of energy, the joule, has units $kg \cdot m^2/s^2$. There is a related system of units: the *CGS* system, which is based on the centimetre, gram, and second. In that system the unit of energy is the ***erg***, which is $1\,erg = 1\,g \cdot cm^2/s^2$. How many ergs are in one joule?
Beginning with one joule, we express kilograms in grams, and metres in centimetres:
$1\,J = 1\,kg \cdot m^2/s^2 = 1\,(1000\,g) \cdot (100\,cm)^2/s^2 = 10^{+7}\,g \cdot cm^2/s^2 = 10^{+7}\,erg$

Exr 4.1.03 The density of Earth's atmosphere (at 101.3 kPa and 15 °C) is approximately $1.225\,kg/m^3$. What is that density measured in grams per litre?
$1.225\,\frac{kg}{m^3} = 1.225\,\frac{1000g}{1000L} = 1.225\,g/L$

4.1.2 Thermal Energy and Temperature

Energy Transferred

Category 1: Using $\Delta E = m\mathscr{C}\,\Delta T$ to find the thermal energy ΔE transferred.

Exr 4.1.04 How many kilojoules of thermal energy must be added to increase the temperature of 3.00 L of water by 2.00 C°?

An increase in the temperature of the object means $\Delta T = +2.00$ C° (notice the sign). One litre of water has a mass of 1 kg (exact), so 3.00 L has a mass of 3.00 kg. Using the relation $\Delta E = m\mathscr{C}\,\Delta T$ with the value $\mathscr{C} = 4\,184$ J/kg · C° for water we get

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.1}$$
$$= (3.00\,kg)(4\,184\,\tfrac{J}{kg\cdot C°})(+2.00\,C°) \tag{4.2}$$
$$= +25\,104\,J = +25.1\,kJ \tag{4.3}$$

The positive sign here signifies that thermal energy was *added* to the object.

Exr 4.1.05 How many kilojoules of thermal energy must be removed from 650 mL of water to decrease its temperature by 16.00 C°?

A decrease in the temperature of the object means $\Delta T = -16.00$ C° (notice the sign). One millilitre of water has a mass of 1 gram, so 650 mL has a mass of 0.650 kg. Using the relation $\Delta E = m\mathscr{C}\,\Delta T$ with the value $\mathscr{C} = 4\,184$ J/kg · C° for water we get

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.4}$$
$$= (0.650\,kg)(4\,184\,\tfrac{J}{kg\cdot C°})(-16.00\,C°) \tag{4.5}$$
$$= -43\,500\,J = -43.5\,kJ \tag{4.6}$$

The negative sign here signifies that thermal energy was *removed* from the object.

Exr 4.1.06 A warm bathtub (172 L) cools by 0.25 C°: how much thermal energy does it lose?

The change in temperature is $\Delta T = -0.25\,\mathrm{C}^\circ$. The mass is $m = 172\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 172\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.7}$$
$$= (172\,\mathrm{kg})(4\,184\,\tfrac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}})(-0.25\,\mathrm{C}^\circ) \tag{4.8}$$
$$= -179\,912\,\mathrm{J} = -180\,\mathrm{kJ} \tag{4.9}$$

Exr 4.1.07 To make a cup of tea we raise 0.53 L by 81 C°: how much thermal energy does this take, in kilojoules?

The change in temperature is $\Delta T = +81\,\mathrm{C}^\circ$. The mass is $m = 0.53\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 0.53\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.10}$$
$$= (0.53\,\mathrm{kg})(4\,184\,\tfrac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}})(+81\,\mathrm{C}^\circ) \tag{4.11}$$
$$= +179\,619\,\mathrm{J} = +180\,\mathrm{kJ} \tag{4.12}$$

Compare this with the previous exercise.

Temperature Change

Category 2: Using $\Delta E = m\mathscr{C}\,\Delta T$ to find the change $\Delta T = T_{\text{final}} - T_{\text{initial}}$ in temperature.

Exr 4.1.08 By how much does the temperature of 3.00 L of water change if we transfer 50.2 kJ into it?

Transferring energy *into* the object means that we are *adding* energy: $\Delta E = +50.2\,\mathrm{kJ} = +50\,200\,\mathrm{J}$ (notice the sign). The mass is $m = 3.00\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 3.00\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.13}$$
$$\Delta T = \Delta E \big/ (m\mathscr{C}) \tag{4.14}$$
$$= (+50\,200\,\mathrm{J}) \Big/ \Big((3.00\,\mathrm{kg})(4\,184\,\tfrac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}})\Big) \tag{4.15}$$
$$= +4.00\,\mathrm{C}^\circ \tag{4.16}$$

The positive sign here signifies that the temperature *increased*.

Exr 4.1.09 By how much does the temperature of 650 mL of water change if we transfer 54.4 kJ out of it?

Transferring energy *out of* the object means that we are *removing* energy: $\Delta E = -54.4\,\mathrm{kJ} = -54\,400\,\mathrm{J}$ (notice the sign). The mass is $m = 0.650\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 0.650\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.17}$$
$$\Delta T = \Delta E \big/ (m\mathscr{C}) \tag{4.18}$$
$$= (-54\,400\,\mathrm{J}) \Big/ \Big((0.650\,\mathrm{kg})(4\,184\,\tfrac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}})\Big) \tag{4.19}$$
$$= -20.0\,\mathrm{C}^\circ \tag{4.20}$$

The negative sign here signifies that the temperature *decreased*.

Exr 4.1.10 By how much does the temperature of 3.00 L of water change if we transfer 753 kJ into it?

Transferring energy *into* the object means that we are *adding* energy: $\Delta E = +753\,\mathrm{kJ} = +753\,000\,\mathrm{J}$ (notice the sign). The mass is $m = 3.00\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 3.00\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.21}$$
$$\Delta T = \Delta E \big/ (m\mathscr{C}) \tag{4.22}$$
$$= (+753\,000\,\mathrm{J}) \Big/ \Big((3.00\,\mathrm{kg})(4\,184\,\tfrac{\mathrm{J}}{\mathrm{kg\cdot C^\circ}})\Big) \tag{4.23}$$
$$= +60.0\,\mathrm{C}^\circ \tag{4.24}$$

The positive sign here signifies that the temperature *increased*.

Exr 4.1.11 By how much does the temperature of 650 mL of water change if we transfer 109 kJ out of it?

Transferring energy *out of* the object means that we are *removing* energy: $\Delta E = -109\,\text{kJ} = -109\,000\,\text{J}$ (notice the sign). The mass is $m = 0.650\,\text{L} \times \frac{1\,\text{kg}}{\text{L}} = 0.650\,\text{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.25}$$
$$\Delta T = \Delta E/(m\mathscr{C}) \tag{4.26}$$
$$= (-109\,000\,\text{J})\Big/\Big((0.650\,\text{kg})(4\,184\,\tfrac{\text{J}}{\text{kg}\cdot\text{C}^\circ})\Big) \tag{4.27}$$
$$= -40.1\,\text{C}^\circ \tag{4.28}$$

The negative sign here signifies that the temperature *decreased*.

Temperature, initial or final

Category 3: Using $\Delta E = m\mathscr{C}\,\Delta T$ to find either the initial temperature T_{initial}, or the final temperature T_{final} from $\Delta T = T_{\text{final}} - T_{\text{initial}}$.

Exr 4.1.12 We add 50.2 kJ of thermal energy to 3.00 L of water that begins at +18 °C. What is its final temperature?

Adding thermal energy means that $E = +50.2\,\text{kJ} = +50\,200\,\text{J}$.
The mass is $m = 3.00\,\text{L} \times \frac{1\,\text{kg}}{\text{L}} = 3.00\,\text{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.29}$$
$$\Delta T = \Delta E/(m\mathscr{C}) \tag{4.30}$$
$$T_{\text{final}} - T_{\text{initial}} = \Delta E/(m\mathscr{C}) \tag{4.31}$$
$$T_{\text{final}} = T_{\text{initial}} + \Delta E/(m\mathscr{C}) \tag{4.32}$$
$$= (+18\,^\circ\text{C}) + \frac{(+50\,200\,\text{J})}{(3.00\,\text{kg})(4\,184\,\frac{\text{J}}{\text{kg}\cdot\text{C}^\circ})} \tag{4.33}$$
$$= (+18\,^\circ\text{C}) + (+4\,\text{C}^\circ) \tag{4.34}$$
$$= +22\,^\circ\text{C} \tag{4.35}$$

Adding thermal energy will increase the temperature, so the final will be higher than the initial.

Exr 4.1.13 730 mL of hot water is allowed to cool. After it has lost 125 kJ of thermal energy the water has cooled to a temperature of +23.1 °C. How hot was it to begin? (What was its initial temperature?)

Losing thermal energy means that $E = -125\,\text{kJ} = -125\,000\,\text{J}$.
The mass is $m = 0.730\,\text{L} \times \frac{1\,\text{kg}}{\text{L}} = 0.730\,\text{kg}$.

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.36}$$
$$\Delta T = \Delta E/(m\mathscr{C}) \tag{4.37}$$
$$T_{\text{final}} - T_{\text{initial}} = \Delta E/(m\mathscr{C}) \tag{4.38}$$
$$T_{\text{initial}} = T_{\text{final}} - \Delta E/(m\mathscr{C}) \tag{4.39}$$
$$= (+23.1\,^\circ\text{C}) - \frac{(-125\,000\,\text{J})}{(0.730\,\text{kg})(4\,184\,\frac{\text{J}}{\text{kg}\cdot\text{C}^\circ})} \tag{4.40}$$
$$= (+23.1\,^\circ\text{C}) - (-40.9\,^\circ\text{C}) \tag{4.41}$$
$$= +64.0\,^\circ\text{C} \tag{4.42}$$

Losing thermal energy will decrease the temperature, so the initial should be higher than the final.

Exr 4.1.14 We remove 301 kJ of thermal energy from 1.80 L of water that begins at +100. °C (it has just finished boiling). What is its final temperature?

Removing thermal energy means that $\Delta E = -301\,\text{kJ} = -301\,000\,\text{J}$.

The mass is $m = 1.80\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 1.80\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\Delta T \tag{4.43}$$
$$\Delta T = \Delta E/(m\mathscr{C}) \tag{4.44}$$
$$T_{\text{final}} - T_{\text{initial}} = \Delta E/(m\mathscr{C}) \tag{4.45}$$
$$T_{\text{final}} = T_{\text{initial}} + \Delta E/(m\mathscr{C}) \tag{4.46}$$
$$= (+100.\,^\circ\mathrm{C}) + \frac{(-301\,000\,\mathrm{J})}{(1.80\,\mathrm{kg})(4\,184\,\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{C}^\circ})} \tag{4.47}$$
$$= (+100\,^\circ\mathrm{C}) + (-40\,^\circ\mathrm{C}) \tag{4.48}$$
$$= +60\,^\circ\mathrm{C} \tag{4.49}$$

Removing thermal energy will decrease the temperature, so the final will be lower than the initial.

Exr 4.1.15 12.5 L of warm water was being heated. After it has gained 1.674 MJ of thermal energy the water has reached a temperature of +92.0 °C. How warm was it to begin? (What was its initial temperature?)

Gaining thermal energy means that $E = +1.674\,\mathrm{MJ} = +1\,674\,000\,\mathrm{J}$.
The mass is $m = 12.5\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 12.5\,\mathrm{kg}$.

$$\Delta E = m\mathscr{C}\Delta T \tag{4.50}$$
$$\Delta T = \Delta E/(m\mathscr{C}) \tag{4.51}$$
$$T_{\text{final}} - T_{\text{initial}} = \Delta E/(m\mathscr{C}) \tag{4.52}$$
$$T_{\text{initial}} = T_{\text{final}} - \Delta E/(m\mathscr{C}) \tag{4.53}$$
$$= (+92.0\,^\circ\mathrm{C}) - \frac{(+1\,674\,000\,\mathrm{J})}{(12.5\,\mathrm{kg})(4\,184\,\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{C}^\circ})} \tag{4.54}$$
$$= (+92.0\,^\circ\mathrm{C}) - (+32.0\,^\circ\mathrm{C}) \tag{4.55}$$
$$= +60.0\,^\circ\mathrm{C} \tag{4.56}$$

Gaining thermal energy will increase the temperature, so the initial should be lower than the final.

Equilibrium Temperature

Category 4: Mixing together two amounts of water (m_A and m_B) at different initial temperatures (T_{Ai} and T_{Bi}), the thermal energy that one gains will equal the thermal energy lost by the other. This is conservation of energy: $\Delta E_A + \Delta E_B = 0$. Once the system has reached equilibrium the combined mass of water will be at a common final temperature. From conservation of energy we can solve for unknown temperatures or masses:

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.57}$$

(You can practice your algebra skills by deriving this equation, if you want to.)

Exercise 4.1.16 We mix together 1.000 L of water at +30 °C with 1.000 L of water at +20 °C. What will be the final temperature of this mixture?

Let's call the warmer water "A" and the cooler water "B". Both masses are $m_A = m_B = 1.00\,\mathrm{L} \times \frac{1\,\mathrm{kg}}{\mathrm{L}} = 1.00\,\mathrm{kg}$. When we mix these volumes of water to total amount of energy in the mixture does not change, so that the energy that A loses as it cools ($\Delta E_A < 0\,\mathrm{J}$) will be the amount that B gains as it warms ($\Delta E_B = -\Delta E_A > 0\,\mathrm{J}$). This is written as

$$\Delta E_A + \Delta E_B = 0\,\mathrm{J} \tag{4.58}$$

To solve for the final temperature, we write-out the energies:

$$\Delta E_A + \Delta E_B = 0\,\mathrm{J} \tag{4.59}$$
$$(m_A\mathscr{C}(T_{\text{final}} - T_{Ai})) + (m_B\mathscr{C}(T_{\text{final}} - T_{Bi})) = 0\,\mathrm{J} \tag{4.60}$$
$$(m_A + m_B)\mathscr{C}\,T_{\text{final}} = (m_A T_{Ai} + m_B T_{Bi})\mathscr{C} \tag{4.61}$$
$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.62}$$

(Notice how the units of energy, and the value of $\mathscr{C}$, are not relevant here – they cancel out of the governing equation.) The final temperature must be between the initial temperatures of the two quantities mixed. Since in this case the masses are equal, the final temperature will be half-way between the two initial temperatures. Substituting our values

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.63}$$

$$= \frac{(1.00\,\text{kg})(30\,^\circ\text{C}) + (1.00\,\text{kg})(20\,^\circ\text{C})}{1.00\,\text{kg} + 1.00\,\text{kg}} \tag{4.64}$$

$$= +25\,^\circ\text{C} \tag{4.65}$$

As expected, this is the average of the two initial values.

EXERCISE 4.1.17 We mix together 800 mL of water at +30 °C with 1.200 L of water at +20 °C. What will be the final temperature of this mixture?

Let's call the warmer water "A" and the cooler water "B". The masses are $m_A = 800\ \text{mL} \times \frac{1\,\text{kg}}{1000\text{mL}} = 0.800\ \text{kg}$, and $m_B = 1.20\,\text{L} \times \frac{1\,\text{kg}}{\text{L}} = 1.20\,\text{kg}$. We will use the equation developed in the previous exercise:

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.66}$$

The final temperature must be between the initial temperatures of the two substances mixed. Since the masses are unequal, the final temperature will be closer to the initial temperature of the portion with greater mass. Substituting our values

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.67}$$

$$= \frac{(0.800\,\text{kg})(30\,^\circ\text{C}) + (1.20\,\text{kg})(20\,^\circ\text{C})}{0.800\,\text{kg} + 1.20\,\text{kg}} \tag{4.68}$$

$$= +24\,^\circ\text{C} \tag{4.69}$$

As expected, the greater mass (which was cooler) dominates the result.

EXERCISE 4.1.18 We mix together 1.200 L of water at +30 °C with 800 mL of water at +20 °C. What will be the final temperature of this mixture?

Let's call the warmer water "A" and the cooler water "B". The masses are $m_A = 1.20\ \text{L} \times \frac{1\,\text{kg}}{\text{L}} = 1.20\ \text{kg}$, and $m_B = 800\ \text{mL} \times \frac{1\,\text{kg}}{1000\text{mL}} = 0.800\ \text{kg}$. The final temperature must be between the initial temperatures of the two substances mixed. Since the masses are unequal, the final temperature will be closer to the initial temperature of the portion with greater mass. Substituting our values

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.70}$$

$$= \frac{(1.20\,\text{kg})(30\,^\circ\text{C}) + (0.800\,\text{kg})(20\,^\circ\text{C})}{1.20\,\text{kg} + 0.800\,\text{kg}} \tag{4.71}$$

$$= +26\,^\circ\text{C} \tag{4.72}$$

As expected, the greater mass (which was warmer) dominates the result.

EXERCISE 4.1.19 We mix together 1.000 L of water at +30 °C with 93 mL of water that was just boiled +100 °C. What will be the final temperature of this mixture?

Let's call the warmer water "A" and the cooler water "B". The masses are $m_A = 1.000\ \text{L} \times \frac{1\,\text{kg}}{\text{L}} = 1.000\ \text{kg}$, and $m_B = 93\ \text{mL} \times \frac{1\,\text{kg}}{1000\text{mL}} = 0.093\ \text{kg}$. The final temperature must be between the initial temperatures of the two substances mixed. Since the masses are unequal, the final temperature will be closer to the initial temperature of the portion with greater mass. Substituting our values

$$T_{\text{final}} = \frac{m_A T_{Ai} + m_B T_{Bi}}{m_A + m_B} \tag{4.73}$$

$$= \frac{(1.000\,\text{kg})(30\,^\circ\text{C}) + (0.093\,\text{kg})(100\,^\circ\text{C})}{1.000\,\text{kg} + 0.093\,\text{kg}} \tag{4.74}$$

$$= +36\,^\circ\text{C} \tag{4.75}$$

As expected, the greater mass (which was cooler) dominates the result – but the large temperature of the smaller mass did shift the temperature significantly.

4.1.3 Power

The relation between power, time and energy is

$$\Delta E = P \times \Delta t \tag{4.76}$$

In the context of thermal energy and its relation to temperature, this means that

$$m\mathscr{C}\,\Delta T = P \times \Delta t \tag{4.77}$$

Here we must be careful to not confuse the change in temperature ΔT and the interval of time Δt.

The rate at which thermal energy is transferred by conduction through a boundary of area A and thickness L is

$$P = \mathscr{K}\frac{A}{L}\left(T_{\text{env}} - T_{\text{sys}}\right) \tag{4.78}$$

where $T_{\text{env}} - T_{\text{sys}}$ is the temperature difference across the boundary (between the system inside the boundary, and the environment outside the boundary). In that expression $\mathscr{K}$ is the *thermal conductivity* of boundary's material, which measures how easily thermal energy is transferred through the material.

Exr 4.1.20 If the temperature of 1 L of water is changing at rate of +1.37 C° every 5.2 s, at what rate is thermal energy being transferred to the water?
The rate of energy transfer is

$$P = m\mathscr{C}\,\tfrac{\Delta T}{\Delta t} = (1\,\text{L} \times \tfrac{1\,\text{kg}}{1\,\text{L}}) \times (4\,184\,\text{J/kg}\cdot\text{C°}) \times \frac{(+1.37\,\text{C°})}{5.2\,\text{s}} = 1.1\,\text{kW} \tag{4.79}$$

Exr 4.1.21 If a 1.200 kW microwave runs for 60 s by what increment would the temperature of a 250 mL cup of water increase?
From the given equation, the temperature change is

$$\Delta T = \frac{P\,\Delta t}{m\mathscr{C}} = \frac{(1\,200\,\text{W})(60\,\text{s})}{(0.250\,\text{kg})(4\,184\,\text{J/kg}\cdot\text{C°})} = +69\,\text{C°} \tag{4.80}$$

Exr 4.1.22 If a 118 mL cup of tea is cooling at a rate of 23 W, how long will it take to cool 3 C°? (Assume that the tea has the same heat capacity as water.)
From the definition of power $\Delta t = \Delta E/P$. Since the tea is cooling $\Delta E < 0\,\text{J}$ and $P < 0\,\text{W}$ (energy is *leaving* the object). Thus the time taken will be

$$\Delta t = \Delta E/P = (m\mathscr{C}\,\Delta T)/P = (118\,\text{g})(4.184\,\text{J/g}\cdot\text{C°})(-3\,\text{C°})/(-23\,\text{W}) = 64.4\,\text{s} \tag{4.81}$$

which is only slightly more than one minute.

4.1.4 Problems

Problem 4.1.01: A large volume of water at 23.2 °C is separated from a large volume of water at 45.8 °C by a steel wall of area 0.0223 m^2 and thickness 3.07 mm. At these temperatures the thermal conductivity of steel is approximately $\mathscr{K} \approx 13.5\,\text{W/m}\cdot\text{K}$.
(a) At what rate is heat flowing through the steel separator?
(b) If we started with identical initial conditions but the steel separator was replaced with a glass separator ($\mathscr{K} \approx 0.96\,\text{W/m}\cdot\text{K}$) of identical size and thickness, what would be the rate?

(a) For the steel separator the rate of heat flow would be

$$P = \mathscr{K}\frac{A}{L}\left(T_{\text{env}} - T_{\text{sys}}\right) \tag{4.82}$$

$$= (13.5\,\text{W/m}\cdot\text{K})\left(\frac{0.0223\,\text{m}^2}{0.00307\,\text{m}}\right)(45.8\,°\text{C} - 23.2\,°\text{C}) \tag{4.83}$$

$$= 2.22\,\text{kW} \tag{4.84}$$

(b) For the glass separator, the calculation would be identical except for the value of the thermal conductivity. $P = 158\,\mathrm{W}$.

PROBLEM 4.1.02: A person, dressed in winter clothes, is standing outside in February in Montreal. Their clothes (and body fat!) have a combined thermal conductivity of $\mathscr{K} = 0.047\,\mathrm{W/m \cdot K}$, and an effective thickness of 37 mm. The air temperature is $-25\,^\circ\mathrm{C}$, and their internal body temperature is $+37\,^\circ\mathrm{C}$. Assuming a surface area of approximately $1.7\,\mathrm{m}^2$, at what rate are they losing thermal energy?

Considering the person's body to be "the system", the rate at which thermal energy is being transferred is

$$P = \mathscr{K}\,\frac{A}{L}\left(T_{\mathrm{env}} - T_{\mathrm{sys}}\right) \tag{4.85}$$

$$= (0.047\,\mathrm{W/m \cdot K})\left(\frac{1.7\,\mathrm{m}^2}{0.037\,\mathrm{m}}\right)\left((-25\,^\circ\mathrm{C}) - (+37\,^\circ\mathrm{C})\right) \tag{4.86}$$

$$= -134\,\mathrm{W} \tag{4.87}$$

The negative sign here indicates that thermal energy is leaving the person.

PROBLEM 4.1.03: How long would it take a 500 W heater to increase the air temperature from $17\,^\circ\mathrm{C}$ to $21\,^\circ\mathrm{C}$ in room that measures 5.05 m by 4.33 m and is 2.25 m tall? (Air has a density $\rho = 1.225\,\mathrm{kg/m^3}$, and its heat capacity is $\mathscr{C}_{\mathrm{air}} = 1.006\,\mathrm{kJ/kg \cdot C^\circ}$.)

To find the time from the power ($\Delta t = \Delta E / P$) we will need to know the total energy required. To find the energy ($\Delta E = m\mathscr{C}\,\Delta T$) we need to know the mass. From the density and the volume of air in the room, the mass is

$$m = \rho V = (1.225\,\mathrm{kg/m^3})(5.05\,\mathrm{m} \times 4.33\,\mathrm{m} \times 2.25\,\mathrm{m}) = 60.27\,\mathrm{kg} \tag{4.88}$$

The energy is

$$\Delta E = (60.27\,\mathrm{kg})(1.006\,\mathrm{kJ/kg \cdot C^\circ})(+4\,\mathrm{C^\circ}) = +242.5\,\mathrm{kJ} \tag{4.89}$$

(Note carefully that this is *kilo*-joules.) Consequently the time required is $\Delta t = (+242.5\,\mathrm{kJ})/(500\,\mathrm{W}) = 485\,\mathrm{s}$, which is approximately eight minutes.

PROBLEM 4.1.04: A 6.117 kg block of steel is placed into a 14.3 L insulated container of water, and then sealed. If the water was initially at $19.5\,^\circ\mathrm{C}$ and the steel was at $95.8\,^\circ\mathrm{C}$, what will be their final equilibrium temperature? (Use $\mathscr{C}_{\mathrm{steel}} = 497\,\mathrm{J/kg \cdot K}$, and assume that very little thermal energy is exchanged with the surroundings.)

From the text we know that the equilibrium temperature of two objects is given by

$$T_{\mathrm{f}} = \frac{m_A \mathscr{C}_A T_{A\mathrm{i}} + m_B \mathscr{C}_B T_{B\mathrm{i}}}{m_A \mathscr{C}_A + m_B \mathscr{C}_B} \tag{4.90}$$

$$= \frac{(6.117\,\mathrm{kg})(497\,\mathrm{J/kg \cdot C^\circ})(95.8\,^\circ\mathrm{C}) + (14.3\,\mathrm{kg})(4\,184\,\mathrm{J/kg \cdot C^\circ})(19.5\,^\circ\mathrm{C})}{(6.117\,\mathrm{kg})(497\,\mathrm{J/kg \cdot C^\circ}) + (14.3\,\mathrm{kg})(4\,184\,\mathrm{J/kg \cdot C^\circ})} \tag{4.91}$$

$$= 23.2\,^\circ\mathrm{C} \tag{4.92}$$

Even though the steel began near water's boiling point its smaller mass and *much* smaller heat capacity resulted in only a slight change in the water's temperature.

PROBLEM 4.1.05: In room that measures 5.05 m by 4.33 m and is 2.25 m tall? the air temperature is $17.0\,^\circ\mathrm{C}$. If a 2.000 L container of water at $42.0\,^\circ\mathrm{C}$ is placed in this room, what would be the equilibrium temperature of the water and the air in the room? (Ignore any transfer of thermal energy with anything else, like the walls, or anything outside; this is just a crude estimate. Use the facts that air has a density $\rho = 1.225\,\mathrm{kg/m^3}$, and its heat capacity is $\mathscr{C}_{\mathrm{air}} = 1.006\,\mathrm{kJ/kg \cdot C^\circ}$.)

From the density and the volume of air in the room, the mass of air in the room is

$$m = \rho V = (1.225\,\mathrm{kg/m^3})(5.05\,\mathrm{m} \times 4.33\,\mathrm{m} \times 2.25\,\mathrm{m}) = (1.225\,\mathrm{kg/m^3})(49.20\,\mathrm{m^3}) = 60.27\,\mathrm{kg} \tag{4.93}$$

(We ignore the small correction of subtracting the volume occupied by the water!) From the text we know that the equilibrium temperature of two objects is given by

$$T_{\mathrm{f}} = \frac{m_A \mathscr{C}_A T_{A\mathrm{i}} + m_B \mathscr{C}_B T_{B\mathrm{i}}}{m_A \mathscr{C}_A + m_B \mathscr{C}_B} \tag{4.94}$$

$$= \frac{(60.27\,\mathrm{kg})(1\,006\,\mathrm{J/kg \cdot C^\circ})(17.0\,^\circ\mathrm{C}) + (2.000\,\mathrm{kg})(4\,184\,\mathrm{J/kg \cdot C^\circ})(42.0\,^\circ\mathrm{C})}{(60.27\,\mathrm{kg})(1\,006\,\mathrm{J/kg \cdot C^\circ}) + (2.000\,\mathrm{kg})(4\,184\,\mathrm{J/kg \cdot C^\circ})} \tag{4.95}$$

$$= 20.0\,^\circ\mathrm{C} \tag{4.96}$$

(In reality, long before this equilibrium would be achieved thermal energy would be exchanged with the walls, and outside.)

4.2 Mechanical Energy

4.2.1 Kinetic Energy

The linear kinetic energy of an object of mass m moving with speed v is

$$K_{\text{lin.}} = \tfrac{1}{2} m v^2 \tag{4.97}$$

When the mass is measured in kilograms and the speed is measured in metres per second, the kinetic energy calculated by this formula is in joules. The linear (or *translational*) speed can be calculated by

$$v = \Delta x / \Delta t \tag{4.98}$$

where Δx is the change in the object's position (measured along a straight line), and Δt is the time taken to travel that distance.

The angular kinetic energy of an object with moment of inertia $\mathscr{I}$ rotating with angular speed ω is

$$K_{\text{ang.}} = \tfrac{1}{2} \mathscr{I} \omega^2 \tag{4.99}$$

When the moment of inertia is measured in kilograms times metres-squared and the angular speed is measured in radians per second, the kinetic energy calculated by this formula is in joules. The angular (or *rotational*) speed can be calculated by

$$\omega = \Delta\theta / \Delta t \tag{4.100}$$

where $\Delta\theta$ is the change in the object's orientation (measured by a change in angle), and Δt is the time taken to travel that distance. The angular change must be measured in ***radians***:

$$2\pi\,\text{rad} = 360^\circ \tag{4.101}$$

Another common unit of rotation is the measure of the number of *revolutions* of the object (where $1\,\text{rev} = 2\pi\,\text{rad}$), and the speed in revolutions per second, or revolutions per minute (rpm):

$$1\,\text{rps} = \frac{1\,\text{rev}}{1\,\text{s}} = \frac{2\pi\,\text{rad}}{1\,\text{s}} \approx 6.23\,\text{rad/s} \tag{4.102}$$

$$1\,\text{rpm} = \frac{1\,\text{rev}}{1\,\text{min}} = \frac{2\pi\,\text{rad}}{60\,\text{s}} \approx 0.1047\,\text{rad/s} \tag{4.103}$$

If an object (or system of objects) has parts moving independently of each other, then the kinetic energy of the system is just the sum of the kinetic energies of the parts. If an object is rotating while it is also moving from place to place, then its kinetic energy is the sum of its linear kinetic energy and its angular kinetic energy.

Linear Kinetic Energy

Exr 4.2.01 What is the kinetic energy of a baseball (mass 153 grams) thrown at a speed of 23 m/s?
$K = \frac{1}{2}(0.153\,\text{kg})(23\,\frac{\text{m}}{\text{s}})^2 = 40\,\text{J}$.

Exr 4.2.02 What is the kinetic energy of a bird (mass 74.8 grams) flying at a speed of 11.6 m/s?
$K = \frac{1}{2}(0.0748\,\text{kg})(11.6\,\frac{\text{m}}{\text{s}})^2 = 5.03\,\text{J}$.

Exr 4.2.03 What is the kinetic energy of a bowling ball (mass 5.17 kg) moving at 2.21 m/s?
$K = \frac{1}{2}(5.17\,\text{kg})(2.21\,\frac{\text{m}}{\text{s}})^2 = 12.6\,\text{J}$.

Exr 4.2.04 What is the kinetic energy of a car (mass 1341 kg) driving at a speed of 10.0 km/h?
The speed is $10.0\,\frac{\text{km}}{\text{h}} = 10.0 \times \frac{1000\text{m}}{3600\text{s}} = 2.78\,\frac{\text{m}}{\text{s}}$. The kinetic energy is $K = \frac{1}{2}(1341\,\text{kg})(2.78\,\frac{\text{m}}{\text{s}})^2 = 5.17\,\text{kJ}$.

Exr 4.2.05 What is the kinetic energy of a car (mass 1341 kg) driving at a speed of 20.0 km/h?
The speed is $20.0\,\frac{\text{km}}{\text{h}} = 20.0 \times \frac{1000\text{m}}{3600\text{s}} = 5.56\,\frac{\text{m}}{\text{s}}$. The kinetic energy is $K = \frac{1}{2}(1341\,\text{kg})(5.56\,\frac{\text{m}}{\text{s}})^2 = 20.7\,\text{kJ}$. (Notice that in comparison to the previous exercise, since the speed is twice as much, the kinetic energy is *four* times as much.)

Exr 4.2.06 What is the kinetic energy of a car (mass 1341 kg) driving at a speed of 100.0 km/h?
The speed is $100.0\,\frac{\text{km}}{\text{h}} = 100.0 \times \frac{1000\text{m}}{3600\text{s}} = 27.8\,\frac{\text{m}}{\text{s}}$. The kinetic energy is $K = \frac{1}{2}(1341\,\text{kg})(27.8\,\frac{\text{m}}{\text{s}})^2 = 517.\,\text{kJ}$.

Exr 4.2.07 At what speed (in km/h) is the kinetic energy of a 1200 kg car equal to the energy (approximately 1 million joules) released by the explosion of one stick of TNT?
Since $K = \frac{1}{2}mv^2$, we have that $v = \sqrt{2K/m}$. Thus $v =$

$\sqrt{2(10^{+6}\mathrm{J})/(1200\,\mathrm{kg})} = 40.8\,\frac{\mathrm{m}}{\mathrm{s}}$. This speed is $v = 40.8\,\frac{\mathrm{m}}{\mathrm{s}} = 40.8\,\frac{\mathrm{m}}{\mathrm{s}} \times \frac{1\,\mathrm{km}}{1000\,\mathrm{m}} \times \frac{3600\,\mathrm{s}}{1\,\mathrm{h}} = 147\,\frac{\mathrm{km}}{\mathrm{h}}$.

Exr 4.2.08 What is the kinetic energy of a person (mass 65.2 kg) walking at a speed of 6.44 km/h?
The speed is $6.44\,\frac{\mathrm{km}}{\mathrm{h}} = 6.44 \times \frac{1000\mathrm{m}}{3600\mathrm{s}} = 1.789\,\frac{\mathrm{m}}{\mathrm{s}}$. The kinetic energy is $K = \frac{1}{2}(65.2\,\mathrm{kg})(1.789\,\frac{\mathrm{m}}{\mathrm{s}})^2 = 100.\,\mathrm{J}$.

Exr 4.2.09 What was the kinetic energy of Usain Bolt (mass 94 kg) sprinting at a speed of 10.0 m/s?
$K = \frac{1}{2}(94\,\mathrm{kg})(10.0\,\frac{\mathrm{m}}{\mathrm{s}})^2 = 4.7\,\mathrm{kJ}$.

Exr 4.2.10 A group of twenty-seven young children are running around, fueled by Halloween candy. In this group the average mass and speed of a child are 37.1 kg and 3.87 m/s. What is the kinetic energy present in this group of children?
On average each child has kinetic energy $K_{\text{one}} = \frac{1}{2}(37.1\,\mathrm{kg})(3.87\,\frac{\mathrm{m}}{\mathrm{s}})^2 = 277.8\,\mathrm{J}$. The group thus has twenty-seven times this much kinetic energy $K_{\text{group}} = 27 \times K_{\text{one}} = 7.50\,\mathrm{kJ}$. (Thermal energy is like this, with each molecule moving rapidly and randomly.)

Angular Kinetic Energy

Exr 4.2.11 An object with moment of inertia $\mathscr{I} = 0.875\,\mathrm{kg\cdot m^2}$ is rotating at 4.00 rad/s. What is its angular kinetic energy?
$K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(0.875\,\mathrm{kg\cdot m^2})(4.00\,\mathrm{rad/s})^2 = 7.00\,\mathrm{J}$.

Exr 4.2.12 An object spinning rapidly at 37.1 rad/s has 1.28 J of angular kinetic energy. What is its moment of inertia?
Since $K = \frac{1}{2}\mathscr{I}\omega^2$ we have that $\mathscr{I} = 2K/\omega^2 = 2(1.28\,\mathrm{J})/(37.1\,\mathrm{rad/s})^2 = 0.001\,860\,\mathrm{kg\cdot m^2}$. (The object's moment of inertia is small, which is why we can get it to spin so rapidly.)

Exr 4.2.13 An object is rotating at 45.00 rpm (revolutions per minute). If its moment of inertia is $\mathscr{I} = 1.333\,\mathrm{kg\cdot m^2}$ what is its rotational kinetic energy?
One revolution is $2\pi\,\mathrm{rad}$, so $\omega = 45.00 \times \frac{\mathrm{rev}}{\mathrm{min}} = 45.00 \times \frac{2\pi\,\mathrm{rad}}{60\,\mathrm{s}} = 4.712\,\mathrm{rad/s}$. Thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(1.333\,\mathrm{kg\cdot m^2})(4.712\,\mathrm{rad/s})^2 = 14.80\,\mathrm{J}$.

Exr 4.2.14 An object with moment of inertia $\mathscr{I} = 0.411\,\mathrm{kg\cdot m^2}$ turns half-way around (180°) in 0.604 s. What is its rotational kinetic energy?
A half-turn 180° is half the circle, so $\Delta\theta = \pi\,\mathrm{rad}$. The angular speed has value $\omega = (\pi\,\mathrm{rad})/(0.604\,\mathrm{s}) = 5.201\,\mathrm{rad/s}$. Thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(0.411\,\mathrm{kg\cdot m^2})(5.201\,\mathrm{rad/s})^2 = 5.56\,\mathrm{J}$.

Exr 4.2.15 A hollow sphere with moment of inertia $\mathscr{I} = 0.03820\,\mathrm{kg\cdot m^2}$ completes 7 rotations every 12 seconds. What is its angular kinetic energy?
Its angular speed, expressed in radians per second, is $\omega = 7\,\mathrm{rev}/12\,\mathrm{s} = 7 \times 2\pi\,\mathrm{rad}/12\,\mathrm{s} = 3.665\,\mathrm{rad/s}$. The angular kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(0.03820\,\mathrm{kg\cdot m^2})(3.665\,\mathrm{rad/s})^2 = 0.257\,\mathrm{J}$.

Exr 4.2.16 A rod, pivoted about its end, with moment of inertia $\mathscr{I} = 0.420\,\mathrm{kg\cdot m^2}$ swings 37° in one third of a second. What is its angular kinetic energy?
The angle is $\Delta\theta = 37° \times \frac{2\pi}{360°} = 0.6458\,\mathrm{rad}$. Its angular speed, expressed in radians per second, is $\omega = 0.6458\,\mathrm{rad}/\frac{1}{3}\,\mathrm{s} = 1.937\,\mathrm{rad/s}$. The angular kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(0.420\,\mathrm{kg\cdot m^2})(1.937\,\mathrm{rad/s})^2 = 0.788\,\mathrm{J}$.

The next few exercises are considered *advanced* in that they require you to calculate the moment to inertia from a formula particular to the object's geometry (*which I will give to you*).

Exercise 4.2.17 A BluRay disk (diameter 12 cm, mass 0.067 kg) is rotating at 810 rev/s. What is its angular kinetic energy? (Ignoring the hole in the middle, the moment of inertia is given by $\mathscr{I} = \frac{1}{2}mR^2$ approximately.)
The moment of inertia of the disc is approximately $\mathscr{I} = \frac{1}{2}mR^2 = \frac{1}{2}(0.067\,\mathrm{kg})(\frac{1}{2} \times 0.12\,\mathrm{m})^2 = 1.206 \times 10^{-4}\,\mathrm{kg\cdot m^2}$.

The angular speed, expressed in radians per second, is $\omega = 810\,\mathrm{rev/s} \times (2\pi\,\mathrm{rad/rev}) = 5\,089\,\mathrm{rad/s}$. The angular kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(1.206 \times 10^{-4}\,\mathrm{kg\cdot m^2})(5\,089\,\mathrm{rad/s})^2 = 1\,562\,\mathrm{J}$.

Exercise 4.2.18 A basketball is spinning at three revolutions per second. The basketball is a hollow sphere of circumference 74.9 cm and mass 0.624 kg. What is its angular kinetic energy? (The moment of inertia of a hollow sphere, about an axis through its center, is $\mathscr{I} = \frac{2}{3}mR^2$. Be careful finding the radius!)
We are given the circumference, so $R = C/2\pi = (0.749\,\mathrm{m})/2\pi = 0.1192\,\mathrm{m}$. The basketball's moment of inertia is thus $\mathscr{I} = \frac{2}{3}mR^2 = \frac{2}{3}(0.624\,\mathrm{kg})(0.1192\,\mathrm{m})^2 = 5.911 \times 10^{-3}\,\mathrm{kg\cdot m^2}$.

The ball's angular speed, expressed in radians per second, is $\omega = 3\,\mathrm{rev/s} \times (2\pi\,\mathrm{rad/rev}) = 18.85\,\mathrm{rad/s}$. The angular kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(5.911 \times 10^{-3}\,\mathrm{kg\cdot m^2})(18.85\,\mathrm{rad/s})^2 = 1.05\,\mathrm{J}$.

Exercise 4.2.19 A rectangular rod (mass 5.18 kg, length 41.1 cm, width 17.0 cm) swings about its end. It swings 25° in 0.25 s. What is its angular kinetic energy? (A rectangular rod of length ℓ and width w has a moment of inertia about an axis at its end given by $\mathscr{I} = m\left(\frac{1}{12}w^2 + \frac{1}{3}\ell^2\right)$.)
The rod's moment of inertia is thus $\mathscr{I} = m\left(\frac{1}{12}w^2 + \frac{1}{3}\ell^2\right) = (0.624\,\mathrm{kg})\left(\frac{1}{12}(0.170\,\mathrm{m})^2 + \frac{1}{3}(0.411\,\mathrm{m})^2\right) = 0.3041\,\mathrm{kg\cdot m^2}$.

The angle is $\Delta\theta = 25° \times \frac{2\pi}{360°} = 0.4363\,\mathrm{rad}$. The rod's angular speed, expressed in radians per second, is $\omega =$

$0.4363\,\mathrm{rad}/0.25\,\mathrm{s} = 1.745\,\mathrm{rad/s}$. The angular kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\,\omega^2 = \frac{1}{2}(0.3041\,\mathrm{kg\cdot m^2})(1.745\,\mathrm{rad/s})^2 = 0.463\,\mathrm{J}$.

4.2.2 Work

When a force $\vec{F}$ is applied to an object, if the object moves a distance $\vec{r}$ along a straight line (its *displacement*), then the ***work*** done to the object is

$$W = F\, r \cos\theta \tag{4.104}$$

where θ is the angle between the force $\vec{F}$ and the displacement $\vec{r}$. This transfer of energy (either into the object or out of the object) changes the kinetic energy of the object

$$\Delta K = W \tag{4.105}$$

where $\Delta K = K_{\text{final}} - K_{\text{initial}}$, and $K = \frac{1}{2} m\, v^2$ is the linear kinetic energy.

When a torque $\vec{\tau}$ is applied to an object, if the object turns through an angle $\Delta\theta$ (its *angular displacement*) about the axis defined by the direction of $\vec{\tau}$, then the work done to the object is

$$W = \tau_z\, \Delta\theta \tag{4.106}$$

This transfer of energy (either into the object or out of the object) changes the kinetic energy of the object

$$\Delta K = W \tag{4.107}$$

where $\Delta K = K_{\text{final}} - K_{\text{initial}}$, and $K = \frac{1}{2} \mathscr{I}\, \omega^2$ is the angular kinetic energy.

EXERCISE 4.2.20 What is the work done by a 5.00 N force applied across 70.0 cm if the force is parallel to the displacement?
When the force is parallel to the displacement $\theta = 0°$. The work done is $W = F\, r \cos\theta = (5.00\,\text{N})(0.700\,\text{m})\cos 0° = +3.50\,\text{J}$.

EXERCISE 4.2.21 What is the work done by a 1.11 N force applied across 1.23 m if the force points opposite the displacement?
When the force is opposite to the displacement $\theta = 180°$. $W = F\, r \cos\theta = (1.11\text{ N})(1.23\text{ m})\cos 180° = -1.37\text{ J}$. The force points opposite the motion, trying to slow or stop the object, so it is *removing* energy from the object $W < 0\,\text{J}$.

EXERCISE 4.2.22 What is the work done by a 32.00 N force applied across 2.75 m if the force points 30° above the direction of the displacement?
$W = F\, r \cos\theta = (32.00\,\text{N})(2.75\,\text{m})\cos 30° = +76.2\,\text{J}$.

EXERCISE 4.2.23 What is the work done by a 42.42 N force applied across 5.05 m if the force points 69° above the direction *opposite* the displacement? (Geometrically, if the object is moving along the $+x$-axis, the force points 69° above the $-x$-axis.)
The angle is $\theta = 180° - 69° = 111°$. $W = F\, r \cos\theta = (42.42\,\text{N})(5.05\,\text{m})\cos 111° = -76.8\,\text{J}$.

EXERCISE 4.2.24 What amount of work needs to be done to a fully loaded shopping cart (39.3 kg) to reduce its speed from 1.72 m/s to 0.23 m/s?
Since $W = \Delta K$, we need to find the change in kinetic energy. $K_{\text{initial}} = \frac{1}{2}(39.3\text{ kg})(1.72\ \frac{\text{m}}{\text{s}})^2 = 58.13\text{ J}$, and $K_{\text{final}} = \frac{1}{2}(39.3\text{ kg})(0.23\ \frac{\text{m}}{\text{s}})^2 = 1.04\text{ J}$. Thus $W = \Delta K = K_{\text{final}} - K_{\text{initial}} = 1.04\,\text{J} - 58.13\,\text{J} = -57.1\,\text{J}$

4.2.3 Potential Energy

"Potential energy" is the term used to describe forms of energy that are not kinetic energy, but that may (through interaction) become kinetic energy.

$$U_{\text{G}} = m g h \tag{4.108}$$

where h is the vertical distance above ($h > 0\,\text{m}$) or below ($h < 0\,\text{m}$) the position *chosen* to be the "zero" gravitational potential.

When a material has its size deformed by an amount ΔL, if the deformation is in its elastic regime the energy stored in the material is given by

$$U_{\text{elastic}} = \tfrac{1}{2} k\, (\Delta L)^2 \tag{4.109}$$

It is important to note that ΔL must measure the deformation from the material's equilibrium (unstressed) size, *not* from some arbitrary "initial" size.

EXR 4.2.25 If a 0.375 kg object falls a distance 22.2 cm, by what amount does its gravitational potential change?
We are told that the object falls, which means $\Delta h < 0\,\text{m}$ and $\Delta U_{\text{G}} < 0\,\text{J}$. Thus $\Delta U_{\text{G}} = (0.375\,\text{kg})(9.81\,\text{m/s}^2)(-0.222\,\text{m}) = -0.817\,\text{J}$.

EXR 4.2.26 Go up. Go up a bit more. Then come down a little. Only $h_f - h_i$. The positions along the way do not matter.

EXR 4.2.27 Stretch a spring.

EXR 4.2.28 Stretch a spring that is already stretched.

4.2.4 Conservation of Energy

The fundamental principle of the Conservation of Energy is that if a collection of objects are interacting in ways that exchange (or transform) energy *only between themselves*, then the total energy of that ***system*** of objects does not change:

$$E_{\text{final}} = E_{\text{initial}} \tag{4.110}$$

This is also written

$$\Delta E_{\text{sys}} = 0\,\text{J} \tag{4.111}$$

When the total energy in the system is written as $E_{\text{sys}} = K + U + E_{\text{Th}}$ (where E_{Th} is the thermal energy generated in the system), conservation of energy can be written as

$$\Delta K + \Delta U + \Delta E_{\text{Th}} = 0\,\text{J} \tag{4.112}$$

4.2.5 Power & Efficiency

Power is the rate with respect to time of energy being transferred or transformed:

$$P = \Delta E / \Delta t \tag{4.113}$$

The units of power are *watts*: $1\,\text{W} = 1\,\text{J/s}$. Another unit of power that is sometimes used in the context of vehicles and machinery is the *horsepower*:

$$1\,\text{hp} = 746\,\text{W} \tag{4.114}$$

If we know the power and the amount of time, then we have the amount of energy that was transferred or transformed:

$$\Delta E = P \times \Delta t \tag{4.115}$$

In the context of electricity, where power is measured in kilowatts and time is measured in hours, the amount of energy is measured in *kilowatt-hours*

$$1\,\text{kWh} = 1\,\text{kW} \times 1\,\text{h} = 1\,000\,\text{W} \times 3\,600\,\text{s} = 3.6\,\text{MJ} \tag{4.116}$$

as an exact number. In the context of food energy, the *food calorie* (or kilocalorie) is meaningful: $1\,\text{Cal} = 4.184\,\text{kJ}$.

Efficiency measures the portion of energy that achieves the purpose. If an amount E_{input} is put towards a task, but only E_{output} is performed in the task then the efficiency is defined to be

$$E_{\text{output}} = \mathscr{E} \times E_{\text{input}} \tag{4.117}$$

For example, when gasoline is burnt in a car's engine to produce motion, only about 30% of the energy produced by the combustion manifests as kinetic energy of the car ($\mathscr{E} \approx 0.30$).

EXERCISE 4.2.29 If an old 60 W incandescent light bulb is left on for one hour, how much energy (in kilojoules) does it dissipate?
$\Delta E = P \times \Delta t = (60\,\text{W}) \times (3600\,\text{s}) = 216\,\text{kJ}$

EXERCISE 4.2.30 If a 15 W LED light bulb is left on for one hour, how much energy (in kilojoules) does it dissipate?
$\Delta E = P \times \Delta t = (15\,\text{W}) \times (3600\,\text{s}) = 54\,\text{kJ}$

EXERCISE 4.2.31 A 1200 W microwave is set to run at full power for 77 s. How much energy (in kilojoules) did it use?
$\Delta E = P \times \Delta t = (1200\,\text{W}) \times (77\,\text{s}) = 92\,\text{kJ}$

EXERCISE 4.2.32 A television is left in its "stand-by" mode for eighteen hours. If that mode uses 24.0 W how much energy (in megajoules) did it use?
$\Delta E = P \times \Delta t = (24.0\,\text{W}) \times (18 \times 3600\,\text{s}) = 1.56\,\text{MJ}$

EXERCISE 4.2.33 Hydro Quebec charges 5.91 cents per kilowatt-hour of energy used. If an old 60W incandescent

light bulb is left on for one whole week, how much will it cost?
Express the energy in kWh: $\Delta E = P \times \Delta t = (.060\,\text{kW}) \times (7 \times 24\,\text{h}) = 10.08\,\text{kWh}$.
The cost is: $(10.08\,\text{kWh}) \times (0.0591\,\frac{\$}{\text{kWh}}) = \$0.60$. (Taxes not included, haha.)

4.2.6 Problems

PROBLEM 4.2.01: A 75.00 kg object moving at $5.165\,\frac{\text{m}}{\text{s}}$ has -400 J of work done to it.
(a) What is its initial kinetic energy?
(b) What will be its final kinetic energy?
(c) What will be its final speed? (Notice that the work done to the object is negative (we've removed energy from it) so its speed will decrease.)
(a) The initial kinetic energy was $K_\text{i} = \frac{1}{2}(75.00\,\text{kg})(5.165\,\frac{\text{m}}{\text{s}})^2 = 1000\,\text{J}$.
(b) The final kinetic energy will be $K_\text{f} = K_\text{i} + \Delta K = K_\text{i} + W = 1000\,\text{J} - 400\,\text{J} = 600\,\text{J}$.
(c) The final speed of the object will be $v_\text{f} = \sqrt{2K_\text{f}/m} = \sqrt{2(600\,\text{J})/(75.00\,\text{kg})} = 4.00\,\frac{\text{m}}{\text{s}}$.

PROBLEM 4.2.02: When one litre of gasoline (primarily octane) is combusted approximately 42 MJ of thermal energy are released. An internal combustion engine is mechanism that converts thermal energy into mechanical energy. Due to the Laws of Thermodynamics (specifically the Law that prevents entropy from decreasing) at most 30% of the thermal energy can be converted into mechanical energy. This mechanical energy is the work that increases the kinetic energy of the car. If 50 mL of gasoline is combusted, what final speed (in km/h) can a 1 723 kg minivan achieve if it began at rest?
If one litre releases 42 MJ, then 50 mL releases $\frac{0.050\,\text{L}}{1.000\,\text{L}} \times 42\,\text{MJ} = 2.100$ MJ. The portion of thermal energy that is converted into mechanical energy is

$$E_\text{output} = \mathscr{E} E_\text{input} = (0.30)(2.100\,\text{MJ}) = 0.630\,\text{MJ}$$

Starting from rest means that $K_\text{i} = 0\,\text{J}$, and $\Delta K = K_\text{f} - K_\text{i} = K_\text{f}$. For that reason

$$v_\text{f} = \sqrt{2K_\text{f}/m} = \sqrt{2(0.630 \times 10^{+6}\,\text{J})/(1\,723\,\text{kg})} = 27.04\,\text{m/s} = 97.4\,\text{km/h}$$

PROBLEM 4.2.03: A car ($m_c = 1\,320$ kg) traveling at 60 km/h comes to a stop. Its kinetic energy is converted by friction into thermal energy in the disc brakes. These are two circular discs steel ($\mathscr{C} = 466\,\frac{\text{J}}{\text{kg}\cdot\text{C}^\circ}$), each of mass $m_d = 9.5$ kg. By what amount does their temperature increase?
The thermal energy gained by the discs equals the kinetic energy lost by the car:

$$\Delta E + \Delta K = 0\,\text{J} \tag{4.118}$$

$$(2m_d)\mathscr{C}\,\Delta T + (0\,\text{J} - \tfrac{1}{2}m_c v^2) = 0\,\text{J} \tag{4.119}$$

$$\Delta T = +\frac{m_c v^2}{2(2m_d)\mathscr{C}} = +\frac{(1320\,\text{kg})(\frac{60}{3.6}\,\frac{\text{m}}{\text{s}})^2}{2(2 \times 9.5\,\text{kg})(466\,\frac{\text{J}}{\text{kg}\cdot\text{C}^\circ})} = +20.7\text{C}^\circ \tag{4.120}$$

The brakes do not gain all of this energy all in an instant. The kinetic energy is transformed at a rate that is almost balanced by the rate at which air flowing past the discs cools them.

PROBLEM 4.2.04: Water falls 15.0 m at a rate of $35.0\,\text{m}^3$/s. Only 60% of that water's kinetic energy can be captured and converted to electrical energy by our generators:
(a) What electrical power can be produced?
(b) If we sell the energy produced at 0.06 \$/kWh, how much will we profit per day?

(a) The process of transfer and transformation of energy begins with the water at the top of the waterfall. After it falls the 15 m its gravitational potential energy (between the water and the Earth) is transformed into its kinetic energy.

$$\Delta K = -\Delta U_\text{G} \tag{4.121}$$

$$\Delta K = -m\,g\,\Delta y \tag{4.122}$$

Since $\Delta y = -15.0$ m this will be positive. What we do not have is the mass. What we do have is the *rate* of volume of water (and thus rate of mass).

$$\frac{m}{\Delta t} = (35.0\,\text{m}^3/\text{s}) \times (1000\,\frac{\text{L}}{\text{m}^3}) \times (1\,\frac{\text{kg}}{\text{L}}) = 35\,000\,\text{kg/s} \tag{4.123}$$

If we divide both sides of the energy equation by the time interval we get the power:

$$P = \frac{\Delta K}{\Delta t} = -(\tfrac{m}{\Delta t})\, g\, \Delta y \tag{4.124}$$

$$P = -(35\,000\,\text{kg/s})(9.81\,\text{N/kg})(-15.0\,\text{m}) = 5\,150\,250\,\text{W} \tag{4.125}$$

But we can only capture 60% of this power:

$$P = 0.60 \times 5\,150\,250\,\text{W} = 3\,090\,150\,\text{W} = 3\,090\,\text{kW} \tag{4.126}$$

(approximately 3 megawatts).

(b) Each day we will profit

$$(0.06\,\$/\text{kWh})(3\,090\,\text{kW})(24\,\text{h}) = \$4\,449.60 \tag{4.127}$$

PROBLEM 4.2.05: How much does it cost to heat a 2.50 litre container of coffee from 4°C to 99°C using a 1200 W microwave? (Assume the coffee is like water with $\mathscr{C} = 4.184$ J/gram·°C, and that Hydro Quebec charges \$0.0591/kWh.)

The amount of heat transferred to the coffee is $Q = m\mathscr{C}\,\Delta T = (2500\,\text{gram})(4.184\,\frac{\text{J}}{\text{gram}\cdot{}^\circ\text{C}})(99\,^\circ\text{C} - 4\,^\circ\text{C}) = 993\,700\,\text{J} = 0.276\,\text{kWh}$. This will cost $0.276\,\text{kWh} \times \$0.0591/\text{kWh} = \$0.0163 = \$0.02$. (Of course we round prices upwards, haha. If we were paying *cash*, we would have to round to the nearest increment of five cents!)

PROBLEM 4.2.06: A person is using a 2200 W heater to keep their apartment warm. After twenty minutes off, it turns on for five minutes, and that repeats. If they keep it running like this how much will it cost them for a 30-day month? (Hydro Quebec charges \$0.0591/kWh.)

The amount of time that the heater is on is 5 min during each 25 min. So the total time the heater is on is $\Delta t = 30\,\text{day} \times 24\,\frac{\text{h}}{\text{day}} \times (\frac{5}{25}) = 144\,\text{h}$. The energy consumed will be $\Delta E = P \times \Delta t = (2.200\,\text{kW}) \times (144\,\text{h}) = 316.8\,\text{kWh}$. This will cost $316.8\,\text{kWh} \times \$0.0591/\text{kWh} = \$18.72$.

PROBLEM 4.2.07: One litre of gasoline, when combusted, can release 42 MJ (megajoules) of thermal energy. If my car used 20 litres of gas to drive to Ottawa, then
(a) How much thermal energy did my car produce over the whole trip?
(b) If the trip took 2.0 hours, what was the rate (in kilowatts) at which thermal energy was produced by the engine?
(c) If only 30% of that thermal energy was converted into mechanical work that actually moved my car, what was this mechanical power output, measured in horsepower?

(a) The thermal energy produced was $E = 20\,\text{L} \times 42\,\frac{\text{MJ}}{\text{L}} = 840\,\text{MJ}$.

(b) The rate of energy production was $P_{\text{thermal}} = E/\Delta t = \frac{840\times10^6\,\text{J}}{2.0\times3600\,\text{s}} = 117\times10^3\,\text{J/s} = 117\,\text{kW}$.

(c) Of that power, only $P_{\text{work}} = 0.30 \times P_{\text{thermal}} = 0.30 \times 117$ kW = 35 kW of power was output as mechanical work. Expressed in horsepower, this is $35\,\text{kW} \times \frac{1\,\text{hp}}{746\,\text{W}} = 47\,\text{hp}$.

PROBLEM 4.2.08: As a car drives down the road, its tires are turning. The total kinetic energy of the car is thus the sum of the linear kinetic energy of the body of the car plus the linear and angular kinetic energies of the tires. A wheel of radius R that is rolling at a speed v is rotating with an angular speed $\omega = v/R$.

The body of a car has mass 1091 kg. Each of its tires (R = 29.4 cm) has mass 13.6 kg and moment of inertia $\mathscr{I} = 0.667\,\text{kg}\cdot\text{m}^2$. If this car is traveling at 55 km/h, what is its total kinetic energy?

The total kinetic energy of the car will be the sum of the kinetic energies of the body and wheels. The body has linear kinetic energy, but each of the wheels have both linear and angular contributions.

The car is traveling with speed $v = 55\,\text{km/h} = 55\,\frac{1000\,\text{m}}{3600\,\text{s}} = 15.28\,\text{m/s}$. The body of the car contributes an amount of linear kinetic energy:

$$K = \tfrac{1}{2}(1\,091\,\text{kg})(15.28\,\text{m/s})^2 = 129.kJ \tag{4.128}$$

Each wheel is traveling at $v = 15.28\,\text{m/s}$, but is also rotating at $\omega = v/R = (15.28\,\text{m/s})/(0.294\,\text{m}) = 52.4\,\text{rad/s}$. Each tire has a kinetic energy

$$K = K_{\text{linear}} + K_{\text{angular}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\mathscr{I}\omega^2 \tag{4.129}$$

$$= \tfrac{1}{2}(13.6\,\text{kg})(15.28\,\text{m/s})^2 + \tfrac{1}{2}(0.667\,\text{kg}\cdot\text{m}^2)(52.4\,\text{rad/s})^2 \tag{4.130}$$

$$= 1.609\,\text{kJ} + 0.916\,\text{kJ} = 2.524\,\text{kJ} \tag{4.131}$$

All four tires thus contribute $4 \times 2.524\,\text{kJ} = 10.1\,\text{kJ}$. The total kinetic energy of the whole car is thus $129.\,\text{kJ} + 10.1\,\text{kJ} = 139.\,\text{kJ}$.

PROBLEM 4.2.09: A steam engine expels 81 J of energy to the environment for every 100 J of internal energy it uses. **(a)** How much work does it generate? **(b)** What is the engine's efficiency?

(a) If 81 J of the 100 J was expelled as waste, then only 100 J − 81 J = 19 J can manifest as work.

(b) The efficiency is $\mathscr{E} = W/E = (19\,\text{J})/(100\,\text{J}) = 0.19 = 19\%$.

PROBLEM 4.2.10: A door is being closed. It measures 2.032 m tall by 76.2 cm wide, and has a mass of 11.340 kg. The door swings 90° in 1.66 s.
(a) What is the door's angular kinetic energy?
(b) A short push was given to make the door close, applied only while it swung through the first 15° of its motion. What force was applied (perpendicular) to the edge of the door?

(a) The moment of inertia of a rectangular object of height y and width x about an axis along the edge of its height is $\mathscr{I} = \frac{1}{3}mx^2$. For this door $\mathscr{I} = \frac{1}{3}(11.340\,\text{kg})(0.762\,\text{m})^2 = 2.195\,\text{kg}\cdot\text{m}^2$.

From the given data the door's angular speed is $\omega = \frac{90°}{1.66\,\text{s}} \times \frac{2\pi\,\text{rad}}{360°} = 0.9463\,\text{rad/s}$. The door's kinetic energy is thus $K = \frac{1}{2}\mathscr{I}\omega^2 = \frac{1}{2}(2.195\,\text{kg}\cdot\text{m}^2)(0.9463\,\text{rad/s})^2 = 0.983\,\text{J}$.

(b) The work done by the push is the change in the door's kinetic energy: $\tau_z\,\Delta\theta = \Delta K$. The angle the door swings during the push is $\Delta\theta = 15° \times \frac{2\pi\,\text{rad}}{360°} = 0.2618\,\text{rad}$. The applied torque is the unknown force times the distance from hinges to the edge (which is just the width). Thus

$$\tau_z\,\Delta\theta = \Delta K \tag{4.132}$$

$$F\,x\,\Delta\theta = K - 0\,\text{J} \tag{4.133}$$

$$F = \frac{K}{x\,\Delta\theta} = \frac{0.983\,\text{J}}{(0.762\,\text{m})(0.2618\,\text{rad})} = 4.93\,\text{N} \tag{4.134}$$

PROBLEM 4.2.11: CHALLENGE: A fan of diameter 45 cm is moving air at 3.8 m/s. What is the power exerted by the fan to move the air at this rate? (The density of air is $\rho_{\text{air}} = 1.225\,\text{kg/m}^3$.) The rate at which mass is being taken into the fan is $\frac{\Delta m}{\Delta t} = \frac{\rho A\,\Delta x}{\Delta t} = \rho A \frac{\Delta x}{\Delta t} = \rho A v = (1.225\,\text{kg/m}^3)(\pi(\frac{0.45\,\text{m}}{2})^2)(3.8\,\text{m/s}) = 0.740\,\text{kg/s}$. The rate at which kinetic energy is being added to the air is thus $P = \frac{\Delta K}{\Delta t} = \frac{1}{2}\frac{\Delta m}{\Delta t}v^2 = \frac{1}{2}(0.740\,\text{kg/s})(3.8\,\text{m/s})^2 = 5.35\,\text{W}$.

4.3 Energy in Biological Contexts

In the context of nutrition the unit of energy is the Calorie:

$$1\,\mathrm{Cal} = 4\,184\,\mathrm{J} \tag{4.135}$$

This is the energy that will increase the temperature of one litre of water by $1\,\mathrm{C}^\circ$ (approximately).

For each litre of oxygen that a person respires metabolic processes release approximately 20 kJ of energy that can be used by the body to perform mechanical work, or to maintain temperature.

A unit of power that is sometimes used is the “horsepower”:

$$1\,\mathrm{hp} = 746\,\mathrm{W} \tag{4.136}$$

This power can be achieved (for very brief intervals) by elite athletes!

4.3.1 Units

Exr 4.3.01 What is 100 Cal expressed in kilowatt-hours?
Converting units: Since $1\,\mathrm{kWh} = 3.6\,\mathrm{MJ}$ we have

$$1\,\mathrm{kWh} = 3.6\,\mathrm{MJ} \times \frac{1\,\mathrm{Cal}}{4\,184\,\mathrm{J}} = 860.42\,\mathrm{Cal} \tag{4.137}$$

Consequently

$$100\,\mathrm{Cal} = 100\,\mathrm{Cal} \times \frac{1\,\mathrm{kWh}}{860.42\,\mathrm{Cal}} = 0.116\,\mathrm{kWh} \tag{4.138}$$

Exr 4.3.02 What is half a horse-power expressed in Calories per minute?
This is a question about a *rate* of energy, a power. The power is $\frac{1}{2}\,\mathrm{hp} = \frac{1}{2} \times 746\,\mathrm{W} = 373\,\mathrm{W}$. Converting units:

$$373\,\mathrm{W} = 373\,\tfrac{\mathrm{J}}{\mathrm{s}} \times \tfrac{1\,\mathrm{Cal}}{4184\,\mathrm{J}} \times \tfrac{60\,\mathrm{s}}{1\,\mathrm{min}} = 5.35\,\mathrm{Cal/min} \tag{4.139}$$

Exr 4.3.03 What is 12.2 mL of Oxygen per second expressed in litres of Oxygen per minute?
Converting units:

$$12.2\,\mathrm{mL/s} = 12.2\,\tfrac{\mathrm{mL}}{\mathrm{s}} \times \tfrac{1\,\mathrm{L}}{1000\,\mathrm{mL}} \times \tfrac{60\,\mathrm{s}}{1\,\mathrm{min}} = 0.732\,\mathrm{L/min} \tag{4.140}$$

Exr 4.3.04 What is the “$1\,\mathrm{L}_{O_2} \to 20\,\mathrm{kJ}$” relation expressed in Calories?
Converting units: The energy produced by the litre of oxygen is $20\,\mathrm{kJ} \times \frac{1\,\mathrm{Cal}}{4.184\,\mathrm{kJ}} = 4.78\,\mathrm{Cal}$. Thus the relation is “$1\,\mathrm{L}_{O_2} \to 4.78\,\mathrm{Cal}$”

4.3.2 Metabolic Energy & Power

Exr 4.3.05 What is the rate of metabolic power output of a sitting person who consumes 0.30 L of oxygen per minute?
The conversion of oxygen consumption rate to power is

$$P = 0.30\,\tfrac{\mathrm{L}}{\mathrm{min}} \times 20\,\tfrac{\mathrm{kJ}}{\mathrm{L}} \times \tfrac{1\,\mathrm{min}}{60\,\mathrm{s}} = 100\,\mathrm{W} \tag{4.141}$$

Exr 4.3.06 What is the rate of oxygen consumption during sleep assuming a metabolic rate of 75 W?
Since the power is $75\,\mathrm{W} = 75\,\mathrm{J/s}$, the corresponding rate of oxygen consumption is
$75\,\frac{\mathrm{J}}{\mathrm{s}} \times \frac{1\,\mathrm{L}}{20\,000\,\mathrm{J}} = 0.003\,750\,\mathrm{L/s} = 3.75\,\mathrm{mL/s} = 0.225\,\mathrm{L/min}$.

Exr 4.3.07 If a person eats 1670 Cal a day, what power (measured in watts) are they generating, on average?

$$1670\,\mathrm{Cal/day} = 1670\,\tfrac{\mathrm{Cal}}{\mathrm{day}} \times \tfrac{4184\,\mathrm{J}}{1\,\mathrm{Cal}} \times \tfrac{1\,\mathrm{day}}{24\times60\times60\,\mathrm{s}} = 80.9\,\mathrm{J/s} = 81\,\mathrm{W} \tag{4.142}$$

Exr 4.3.08 If a person's metabolic rate is 75 W while they are sleeping (for eight hours), but their average metabolic rate over a 24 h period is 81 W, then what is their average metabolic rate while they are awake?
If they sleep for 8 h and are awake for 16 h, then the person's average power is

$$P_{\text{avg}} = \frac{(8\,\text{h}\cdot P_{\text{sleep}}) + (16\,\text{h}\cdot P_{\text{awake}})}{8\,\text{h} + 16\,\text{h}} \tag{4.143}$$

. Solving this for the power while awake, we get

$$P_{\text{awake}} = \frac{1}{16\,\text{h}} \times \left((24\,\text{h}\cdot P_{\text{avg}}) - (8\,\text{h}\cdot P_{\text{sleep}})\right) \tag{4.144}$$

$$= \frac{1}{16\,\text{h}} \times ((24\,\text{h}\cdot 81\,\text{W}) - (8\,\text{h}\cdot 75\,\text{W})) = 84\,\text{W} \tag{4.145}$$

(We note that the average is between the sleeping and awake rates, as it should be.)

4.3.3 Biomechanical Energy, Work & Power

Exercise 4.3.09 While a person is walking one of their legs swings forward through an angle of 30° in 0.75 s. If their leg has a moment of inertia $\mathscr{I} = 1.83\,\text{kg}\cdot\text{m}^2$, what is the angular kinetic energy of their leg?
To calculate the angular kinetic energy we need the angular speed of their leg, expressed in radians per second. Since $30° \times \frac{2\pi\,\text{rad}}{360°} = 0.524\,\text{rad}$, the angular speed is $\omega = \frac{\Delta\theta}{\Delta t} = \frac{0.524\,\text{rad}}{0.75\,\text{s}} = 0.699\,\text{rad/s}$. Thus the kinetic energy of their leg is

$$K = \tfrac{1}{2}\mathscr{I}\omega^2 = \tfrac{1}{2}(1.83\,\text{kg}\cdot\text{m}^2)(0.699\,\text{rad/s})^2 = 0.447\,\text{J} \tag{4.146}$$

Exercise 4.3.10 A person is waving their arm through an angle of 42°, one side to the other, four times in 1.08 s. If their arm has a moment of inertia $\mathscr{I} = 0.583\,\text{kg}\cdot\text{m}^2$, what is the angular kinetic energy of their arm?
To calculate the angular kinetic energy we need the angular speed of their arm, expressed in radians per second. Since $42° \times \frac{2\pi\,\text{rad}}{360°} = 0.733\,\text{rad}$, the angular speed is $\omega = \frac{\Delta\theta}{\Delta t} = \frac{0.733\,\text{rad}}{1.08\,\text{s}/4} = 2.715\,\text{rad/s}$. Thus the kinetic energy of their leg is

$$K = \tfrac{1}{2}\mathscr{I}\omega^2 = \tfrac{1}{2}(0.583\,\text{kg}\cdot\text{m}^2)(2.715\,\text{rad/s})^2 = 2.15\,\text{J} \tag{4.147}$$

Exercise 4.3.11 A person opening a door does 1.305 J work to it (changing its motion from being at rest to swinging open). Their upper arm rotates about their shoulder 107° while pulling on the door. If the muscles in the shoulder complex can be modeled as a single force acting along a line 20.8 mm from the shoulder joint, what is the tension in these muscles?
The work done to the door (1.305 J) relates to the torque generated about their shoulder by $W = \tau\,\Delta\theta$, and the torque depends upon the tension T in the muscles by $\tau = Td$. Thus $T = W/(d\,\Delta\theta)$.

Expressed in radians $\Delta\theta = 107° \times \frac{2\pi\,\text{rad}}{360°} = 1.866\,\text{rad}$. Thus the tension is

$$T = \frac{W}{d\,\Delta\theta} = \frac{1.305\,\text{J}}{(0.0208\,\text{m})(1.866\,\text{rad})} = 33.6\,\text{N} \tag{4.148}$$

Exercise 4.3.12 A person performs 0.103 kWh of work during their day. If the efficiency of the food-to-work process in their body is only 18%, how many Calories did their body consume to perform this work?
Expressing the work performed in Calories: $W_{\text{out}} = 0.103\,\text{kWh} \times \frac{3.6\,\text{MJ}}{1\,\text{kWh}} \times \frac{1\,\text{Cal}}{4.184\,\text{kJ}} = 88.6\,\text{Cal}$. If this was only 18% of the energy consumed, then $E_{\text{input}} = W_{\text{out}}/\mathscr{E} = 88.6\,\text{Cal}/(0.18) = 492\,\text{Cal}$.

4.3.4 Problems

Exercise 4.3.13 A person who is standing upright lifts one of their arms (3.5 kg) from straight at their side to straight over their head. During this motion the center of mass of this segment is raised 55 cm. If they repeat this motion 40 times, and the efficiency of their internal energy conversion is 18%, how many calories will this require?

Raising 3.5 kg a height 0.55 m requires $W = \Delta U_{\text{G}} = mg\,\Delta h = (3.5\,\text{kg})(9.81\,\frac{\text{N}}{\text{kg}})(0.55\,\text{m}) = 18.88\,\text{J}$. Repeating this 40 times requires $30 \times 18.88\,\text{J} = 755.37\,\text{J}$. If that was only 18% of the internally-generated energy, then

$$E_{\text{in}} = W/\mathscr{E} = (755.37\,\text{J})/0.18 = 4\,196.5\,\text{J} \tag{4.149}$$

which is $4\,196.5\,\text{J} \times \frac{1\,\text{Cal}}{4\,184\,\text{J}} = 1.0\,\text{Cal}$.

Problem 4.3.01: One serving of Pad Thai contains about 573 Calories of food energy (1 Calorie = 4.184 kJ). If you have a body mass of 65 kg, when you walk at 9 km/h you "burn" 9.55 Calories per minute.
(a) What is the rate of energy expenditure, measured in watts?
(b) At this rate how long (in hours) would it take to "burn off" your Pad Thai?

(a) The power of 9.55 Calories per minute equals $9.55\,\frac{\text{Cal}}{\text{min}} \times \frac{4184.\text{J}}{\text{Cal}} \times \frac{1\,\text{min}}{60\,\text{s}} = 666\,\text{W}$. (Note: This is almost one horsepower – this seems enormous! But remember that this is the *internally* generated energy, and only a *fraction* of this will manifest *externally* as the mechanical work performed to move your body around.)

(b) The energy to be developed is $573\,\text{Cal} = 2.40 \times 10^6$ J.
The time this will take is $\Delta t = \Delta E/P = \frac{2.40\times10^6\,\text{J}}{666\,\text{W}} = 3.60 \times 10^3$ s, which is one hour.

Problem 4.3.02: Assuming that muscles convert food energy into mechanical energy with an efficiency of 22%, how much food energy is converted by an 80-kg man climbing a vertical distance of 15m? Express your answer in kilojoules, and in food Calories.

The mechanical work he does changes his gravitational potential energy:

$$W = \Delta U_{\text{G}} = mg\,\Delta h = (80\,\text{kg})(9.81\,\frac{\text{N}}{\text{kg}})(+15\,\text{m}) = +11\,772\,\text{J} \tag{4.150}$$

This work is only 22% of the food energy he consumed. The full 100% is $(11\,772\,\text{J})/0.22 = 53\,509\,\text{J} = 53.5\,\text{kJ}$. Converting this to food Calories gives $53.5\,\text{kJ} \times \frac{1\,\text{Cal}}{4.184\,\text{kJ}} = 12.8\,\text{Cal}$.

Problem 4.3.03: A grocery store worker is placing pop bottles on a shelf. Every 24 s they place ten 2 L bottles on the shelf, raising them 72 cm from the delivery pallet to the shelf. If they continue this job, at this rate, for 13 minutes:
(a) What is the total work done?
(b) How many Calories did they "burn"? (Assume $\mathscr{E} = 0.20$.)

(a) The work done to each bottle equals the change in its gravitational potential energy: $W = \Delta U_{\text{G}} = mg\,\Delta h = (2\,\text{kg})(9.81\,\frac{\text{N}}{\text{kg}})(+0.72\,\text{m}) = +14.13\,\text{J}$.

The total number of bottles that they lifted was $\#\text{bottles} = \text{rate} \times \text{time} = \frac{10\,\text{bottles}}{24\,\text{s}} \times (13 \times 60\,\text{s}) = 325\,\text{bottles}$.

Thus the total work done was $W = 325 \times (+14.13\,\text{J}) = +4\,591\,\text{J}$.

(b) Since $\mathscr{E} = W/E_{\text{input}}$, the input energy (the amount of chemical food energy that they "burned") was

$$E_{\text{input}} = \frac{W}{\mathscr{E}} = \frac{4\,591\,\text{J}}{0.20} \times \frac{1\,\text{Cal}}{4\,184\,\text{J}} = 5.41\,\text{Cal} \tag{4.151}$$

Problem 4.3.04: An 80-kg man runs up stairs, ascending 6.0 m in 8.0 s. What is his power output in kilowatts, and in horsepower?

The work done by the man is the change in his gravitational potential energy:

$$\Delta U_{\text{G}} = mg\,\Delta h = (80\,\text{kg})(9.81\,\frac{\text{N}}{\text{kg}})(+6.0\,\text{m}) = +4\,709\,\text{J} = 4.709\,\text{kJ} \tag{4.152}$$

This work was performed at a rate of

$$P = \frac{W}{\Delta t} = \frac{4.709\,\text{kJ}}{8.0\,\text{s}} = 0.589\,\text{kW} \tag{4.153}$$

Expressed in horsepower, this is $589\,\text{W} \times \frac{1\,\text{hp}}{746\,\text{W}} = 0.789\,\text{hp}$.

Problem 4.3.05: The chemical process that drives muscles can produce about 20 kJ of mechanical energy for each litre of oxygen a human respires. If a sprinter consumes 4.1 L/min of oxygen, what is their maximum possible power output, in watts, and in Calories per minute?

The rate of oxygen volume consumption is 4.1 L/min = 4.1 L/60 s = 0.068 333 L/s. Applying the conversion of oxygen volume to energy produced, the power generated is

$$P = 0.068\,333\,\tfrac{\text{L}}{\text{s}} \times 20\,\tfrac{\text{kJ}}{\text{L}} = 1.367\,\tfrac{\text{kJ}}{\text{s}} = 1\,367\,\tfrac{\text{J}}{\text{s}} = 1\,367\,\text{W} \tag{4.154}$$

That's almost two horsepower! Obviously this is not a power that can be sustained for very long – seconds at most – and only by Olympic-level athletes. Since 1 Calorie = 4 184 J, this power is also $1\,367\,\tfrac{\text{J}}{\text{s}} \times \tfrac{1\,\text{Cal}}{4\,184\,\text{J}} \times \tfrac{60\,\text{s}}{1\,\text{min}} = 19.603\,\text{Cal/min}$.

PROBLEM 4.3.06: A 64kg woman is slowly descending the stairs, traveling 16.5m downwards in 37s. **(a)** At what rate must her body dissipate gravitational energy so that so descends at a constant rate? **(b)** If all of that energy was converted into thermal energy in her body, by what amount would her temperature increase?

(a) The power is the change divided by the time. The change in gravitational energy is

$$\Delta U_\text{G} = mg\,\Delta h = (64\,\text{kg})(9.81\,\tfrac{\text{N}}{\text{kg}})(-16.5\,\text{m}) = -10\,359\,\text{J} \tag{4.155}$$

The dissipation power must be

$$P = \frac{-10\,359\,\text{J}}{37\,\text{s}} = -280\,\text{W} \tag{4.156}$$

(b) The gravitational energy that was dissipated becomes thermal energy in her body:

$$\Delta E = m\mathscr{C}\,\Delta T \tag{4.157}$$
$$\Delta T = \Delta E / m\mathscr{C} \tag{4.158}$$
$$= (+10\,359\,\text{J})/(64\,\text{kg} \times 4\,184\,\tfrac{\text{J}}{\text{kg}\cdot\text{C}^\circ}) \tag{4.159}$$
$$= +0.0387\,\text{C}^\circ \approx +0.04\,\text{C}^\circ \tag{4.160}$$

This is barely noticeable, and is easily removed by respiration and thermal radiation.

PROBLEM 4.3.07: In the video "Olympic Cyclist vs Toaster", Robert Förstemann (Germany) sustains a power output of 700W for one minute, approximately. (Notice: That is almost one horse-power!) **(a)** What was his total energy output, measured in kilowatt-hours? **(b)** If that output was only 22% of the chemical energy consumed by his muscles, how many Calories did that effort require? **(c)** Consequently, how much oxygen did he need to respire?

(a) The energy output was $\Delta E = P \times \Delta t = (700\,\text{W})(60\,\text{s}) = 42\,000\,\text{J} \times \tfrac{1\,\text{kWh}}{3.6\times10^{+6}\,\text{J}} = 0.012\,\text{kWh}$.

(b) That amount of energy output was $42\,000\,\text{J} \times \tfrac{1\,\text{Cal}}{4\,184\,\text{J}} = 10.04\,\text{Cal}$. If that was only 22%, then he actually produced $E = \tfrac{1}{0.22} \times 10.04\,\text{Cal} = 45.63\,\text{Cal}$.

(c) The relation between oxygen respired and metabolic energy is $\tfrac{1\,\text{L}}{20\,000\,\text{J}} = \tfrac{1\,\text{L}}{4.780\,\text{Cal}}$, he respired $\tfrac{1\,\text{L}}{4.780\,\text{Cal}} \times 45.63\,\text{Cal} = 9.55\,\text{L}$.

PROBLEM 4.3.08: While a person is walking one of their leg swings forward through an angle of 30° in 0.75 s. Their leg has a moment of inertia $\mathscr{I} = 1.83\,\text{kg}\cdot\text{m}^2$. At the beginning of the swing their leg has $\omega = 0\,\text{rad/s}$, and the maximum occurs half-way through the swing, where ω equals *twice* the average angular speed across the whole motion. Assuming the moment arm between the muscles and the hip joint is approximately 5 cm, what it the tension in the muscles during this motion?

The tension in the muscle will exert a torque about the hip on the leg. This torque will do (angular) work to the leg as it swings. This work will change the leg's angular speed.

Since $30^\circ \times \tfrac{2\pi\,\text{rad}}{360^\circ} = 0.524\,\text{rad}$, the average angular speed across the motion is $\omega = \tfrac{\Delta\theta}{\Delta t} = \tfrac{0.524\,\text{rad}}{0.75\,\text{s}} = 0.699\,\text{rad/s}$. The maximum is thus $\omega = 1.398\,\text{rad/s}$.

Since the leg began at rest the work done to accelerate it from rest to that maximum was

$$W = \Delta K_\text{angular} \tag{4.161}$$
$$W = \tfrac{1}{2}\mathscr{I}\,\omega_\text{f}^2 - 0\,\text{J} \tag{4.162}$$
$$= \tfrac{1}{2}(1.83\,\text{kg}\cdot\text{m}^2)(1.398\,\text{rad/s})^2 = +1.788\,\text{J} \tag{4.163}$$

This work equals the torque times the angle to that half-way position.

$$\tau_z\,\Delta\theta = \Delta K_{\text{angular}} \tag{4.164}$$
$$\tau_z = \Delta K_{\text{angular}}/\Delta\theta \tag{4.165}$$
$$= (1.788\,\text{J})/(\tfrac{1}{2}\times 0.524\,\text{rad}) = 6.825\,\text{N·m} \tag{4.166}$$

And this torque is the tension in the muscle times the moment arm

$$T\,d = \tau_z \tag{4.167}$$
$$T = \tau_z/d = (6.825\,\text{N·m})/(0.05\,\text{m}) = 137\,\text{N} \tag{4.168}$$

(Given what we learned about biomechanics in the chapter on Torque, we see that this is a very modest amount of tension, which makes sense since walking is not very strenuous.)

Problem 4.3.09: A person nodding their head ("yes") moves their head 15° up-and-down five times in 1.83 s. For this rotation their head has a moment of inertia $\mathscr{I} = 0.0847\,\text{kg}\cdot\text{m}^2$. The muscles at the back of their neck act along a line 43.8 mm from the joint at the base of their skull. What is the tension in these muscles? (Note: At the beginning of the "nod" head leg has $\omega = 0\,\text{rad/s}$, and the maximum occurs half-way through the motion, where ω equals *twice* the average angular speed across the whole motion.)

The tension in the muscle will exert a torque about the cervical on the head. This torque will do (angular) work to the head as it nods. This work will change the head's angular speed.

Since $15° \times \frac{2\pi\,\text{rad}}{360°} = 0.262\,\text{rad}$, and one "nod" is (up-and-down) twice this, the average angular speed across the motion is $\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\times 0.262\,\text{rad}}{1.83\,\text{s}/5} = 1.431\,\text{rad/s}$. The maximum is thus $\omega = 2.861\,\text{rad/s}$.

Since their head began at rest the work done to accelerate it from rest to that maximum was

$$W = \Delta K_{\text{angular}} \tag{4.169}$$
$$W = \tfrac{1}{2}\mathscr{I}\,\omega_{\text{f}}^2 - 0\,\text{J} \tag{4.170}$$
$$= \tfrac{1}{2}(0.0847\,\text{kg}\cdot\text{m}^2)(2.861\,\text{rad/s})^2 = +0.347\,\text{J} \tag{4.171}$$

This work equals the torque times the angle to that half-way position.

$$\tau_z\,\Delta\theta = \Delta K_{\text{angular}} \tag{4.172}$$
$$\tau_z = \Delta K_{\text{angular}}/\Delta\theta \tag{4.173}$$
$$= (0.347\,\text{J})/(\tfrac{1}{2}\times 0.262\,\text{rad}) = 2.65\,\text{N·m} \tag{4.174}$$

And this torque is the tension in the muscle times the moment arm

$$T\,d = \tau_z \tag{4.175}$$
$$T = \tau_z/d = (2.65\,\text{N·m})/(0.0438\,\text{m}) = 60.4\,\text{N} \tag{4.176}$$

Chapter 5

Waves

5.1 Periodic Waves

5.1.1 Period & Frequency

In these exercises we will be determining the period T (the amount of time between repetitions) and the frequency f (the rate of repetition) of various phenomena. The frequency f of a repeating event is defined to be the ratio of the number of repetitions to the time taken for those repetitions to occur, and is measured in hertz:

$$1\,\text{hertz} = 1\,\frac{\text{repetition}}{\text{second}} \tag{5.1}$$
$$1\,\text{Hz} = 1\,\frac{\text{rep}}{\text{s}} \tag{5.2}$$

(Note that "repetition" is not an SI unit, but is a place-holder or reminder. When calculating it does not contribute anything except to help keep track of consistency. Strictly speaking $1\text{ Hz} = 1\text{ s}^{-1}$.) Because of their definitions they relate: $f = 1/T$. So we can also write the units of period as "seconds per repetition".

Exr 5.1.01 If a person's heart rate is 70 beats per minute, what are the period and frequency of that oscillation?
The frequency f of a repeating event is defined to be the ratio of the number of repetitions to the time taken for those repetitions to occur. In this problem

$$f = \frac{70\text{ rep}}{60\text{ s}} = 1.17\text{ Hz} \tag{5.3}$$

The period T of a repeating event is defined to be the time taken per repetition

$$T = \frac{60\text{ s}}{70\text{ rep}} = 0.857\text{ s} \tag{5.4}$$

Remember that $f = 1/T$ can be used to get one from the other.

Exr 5.1.02 The human heart rate can fall in the range from 60 beats per minute (very relaxed) to 120 beats per minute (intense exercise). To what range of frequencies does this correspond?
The range of heart rates mentioned for humans corresponds to frequencies in the range $(60\text{ rep})/(60\text{ s}) = 1\text{ Hz}$ to $(120\text{ rep})/(60\text{ s}) = 2\text{ Hz}$.

Exr 5.1.03 An adult human at rest has a respiration rate between 12 to 16 times a minute. To what range of frequencies does this correspond?
The range of respiration rates mentioned corresponds to frequencies in the range $(12\text{ rep})/(60\text{ s}) = 0.20\text{ Hz}$ to $(16\text{ rep})/(60\text{ s}) = 0.27\text{ Hz}$.

Exr 5.1.04 During exercise a person is respiring at a rate of 40 breaths per minute. What is period and frequency of this rate?
This rate of respiration has period and frequency $T = (60\text{ s})/(40\text{ rep}) = 1.5\text{ s}$ and $f = (40\text{ rep})/(60\text{ s}) = 0.67\text{ Hz}$, respectively.

Exr 5.1.05 The average computer user can type about 190 characters per minute, while professional typists can type as fast as 360 characters per minute. What are the frequency of keys per second for these two keyboardists?
For the average user $f = (190\text{ rep})/(60\text{ s}) = 3.17\text{ Hz}$ while for the professional $f = (360\text{ rep})/(60\text{ s}) = 6.00\text{ Hz}$.

Exr 5.1.06 Players of video games sometimes need to perform "button mashing" when they repeatedly press a control a rapidly as possible. A player pressed a button 191 times in 12 seconds. What are the period and frequency of this rate of button mashing?
This rate of button mashing has period and frequency $T = (12\text{ s})/(191\text{ rep}) = 0.0628\text{ s}$ and $f = (191\text{ rep})/(12\text{ s}) = 15.92\text{ Hz}$, respectively.

Exr 5.1.07 The average male voice has an oscillation around 130 Hz. A bass singer usually has a low note around 82.4 Hz. One of the lowest bass singers, a fellow by the name Tim Storms, can sing notes lower than B0 (30.87 Hz)! What are the periods of oscillation of the vocal chords of men making these three notes?
For each of these three cases we need to use $T = 1/f$:
$T = 1/f = 1/(130\text{ Hz}) = 0.007692\text{ s} = 7.69\text{ ms}$.
$T = 1/f = 1/(82.4\text{ Hz}) = 0.01214\text{ s} = 12.1\text{ ms}$.
$T = 1/f = 1/(30.87\text{ Hz}) = 0.03239\text{ s} = 32.4\text{ ms}$ (four times slower than speaking!).

Exr 5.1.08 The average female voice has an oscillation around 210 Hz. A soprano singer usually has a high note around 650 Hz. Above the soprano range is the so-called

whistle register. Mariah Carey was once recorded hitting G7 (3.136 kHz)! What are the periods of oscillation of the vocal chords of women making these three notes?
For each of these three cases we need to use $T = 1/f$:
$T = 1/f = 1/(210\,\text{Hz}) = 0.004762\,\text{s} = 4.76\,\text{ms}$.
$T = 1/f = 1/(650\,\text{Hz}) = 0.001538\,\text{s} = 1.54\,\text{ms}$.
$T = 1/f = 1/(3\,136\,\text{Hz}) = 0.0003189\,\text{s} = 0.319\,\text{ms}$.

It is interesting to note that the biomechanics of how notes in the whistle register are produced is not clearly understood! It is also not exclusive to female vocalists, as there are some male vocalists with this range. Also these notes are easily produced by young children.

Exr 5.1.09 The heart of a hummingbird can beat as fast as 1260 times a minute. What is this rate measured in hertz?
The frequency of a repeating event is defined to be "number of repetitions / time". Since 1 minute is 60 seconds, the frequency is $f = (1260\,\text{rep})/(60\,\text{s}) = 21\,\text{Hz}$.

Exr 5.1.10 The wings of a hummingbird can flap as fast as 10 000 times in three minutes! What is this rate measured in hertz?
The frequency of a repeating event is defined to be "number of repetitions / time". Since 1 minute is 60 seconds, the frequency is $f = (10\,000\,\text{rep})/(180\,\text{s}) = 56\,\text{Hz}$.

Exr 5.1.11 The musical-sounding notes produced by birds, *birdsong*, falls in the range 1 kHz to 8 kHz. (Compare this with the whistle register of human singing.) What is the range of the periods of oscillation of the vocal chords of birds making notes in this range?
To calculate the extremes of the range we need to use $T = 1/f$:
$T = 1/f = 1/(1\,000\,\text{Hz}) = 0.00100\,\text{s} = 1.00\,\text{ms}$.
$T = 1/f = 1/(8\,000\,\text{Hz}) = 0.000125\,\text{s} = 0.125\,\text{ms}$.

Exr 5.1.12 Lower-frequency sounds travel further than high-frequency sounds. (This is due to the phenomena of *attenuation* that we will study later.) Elephants communicate using some frequencies below what humans can hear. The tones they produce fall in the range of 5 Hz to 30 Hz. What is the range of the periods of oscillation corresponding to notes in this range?
To calculate the extremes of the range we need to use $T = 1/f$:
$T = 1/f = 1/(5\,\text{Hz}) = 0.200\,\text{s}$.
$T = 1/f = 1/(30\,\text{Hz}) = 0.0333\,\text{s} = 33.3\,\text{ms}$.

Note that these sounds are not produced by only the vocal chords of the elephants, but by coupled resonance of the air mass in their trunks.

5.1.2 The Fundamental Relationship for Periodic Waves

These exercises will use the fundamental relationship for periodic waves

$$\lambda = v/f \tag{5.5}$$

to relate Frequency (f), Wavelength (λ) and Wavespeed (v). The SI units to use for these quantities are hertz (Hz) for frequency, metres (m) for wavelength, and metres per second (m/s) for wavespeed.

Exr 5.1.13 What is the speed of a periodic wave whose frequency and wavelength are 500 Hz and 0.5 m respectively?
Here $v = \lambda f = (0.5\,\text{m})(500\,\text{Hz}) = 250\,\text{m/s}$.

Exr 5.1.14 What is the wavespeed on a steel cable if a periodic wave of frequency 75 Hz has wavelength 1.333 m?

Here $v = \lambda f = (1.333\,\text{m})(75\,\text{Hz}) = 100\,\text{m/s}$.

Exr 5.1.15 What is the speed of an ultrasonic wave in flesh whose frequency and wavelength are 2.20 MHz and 705 µm respectively?
$v = \lambda f = (705\ \mu\text{m})(2.20\ \text{MHz}) = (705 \times 10^{-6}\text{m})(2.20 \times 10^{+6}\text{Hz}) = 705 \times 2.20\,\text{m/s} = 1\,551\,\text{m/s}$.

Exr 5.1.16 What is the speed of a ultrasonic wave in air whose frequency and wavelength are 45.0 kHz and 7.63 mm respectively?
$v = \lambda f = (7.63\ \text{mm})(45.0\ \text{kHz}) = (7.63 \times 10^{-3}\text{m})(45.0 \times 10^{+3}\text{Hz}) = 7.63 \times 45.0\,\text{m/s} = 343\,\text{m/s}$.

Exr 5.1.17 What is the wavelength of a periodic wave on a stretched spring whose wavespeed and period of oscillation are are 75 m/s and 0.005 s respectively?
Since $f = 1/T$ we have $\lambda = v/f = vT$. Here $\lambda = vT = (75\,\text{m/s})(0.005\,\text{s}) = 0.375\,\text{m}$.

Exr 5.1.18 Electrical power is delivered by oscillating voltages of frequency 60.0 Hz. These electrical oscillations can sometimes cause mechanical oscillations of equal frequency, which then become sound in the surrounding air. Since the wavespeed in air is 343 m/s what is the wavelength of these sounds?
Here $\lambda = v/f = (343\,\text{m/s})/(60.0\,\text{Hz}) = 5.72\,\text{m}$.

Exr 5.1.19 What is the wavelength of an underwater sound wave whose wavespeed and frequency of oscillation are are 1480 m/s and 5.90 kHz respectively?
Here $\lambda = v/f = (1480\,\text{m/s})/(5\,900\,\text{Hz}) = 0.251\,\text{m}$.

Exr 5.1.20 Some ultrasonic waves traveling through fat tissue have wavespeed and frequency of oscillation 1450 m/s and 1.55 MHz respectively. What is the wavelength of these waves?
$\lambda = v/f = (1450\ \text{m/s})/(1.55 \times 10^{+6}\ \text{Hz}) = 9.35 \times 10^{-4}\ \text{m} = 0.935\,\text{mm}$.

Exr 5.1.21 What is the frequency of a periodic wave whose wavespeed and wavelength are 120 m/s and 30 cm respectively?
Since $\lambda = v/f$ we can get that $f = v/\lambda$.
Here $f = v/\lambda = (120\,\text{m/s})/(0.30\,\text{m}) = 400\,\text{Hz}$.

Exr 5.1.22 What is the frequency of a wave on a stretched spring whose wavespeed and wavelength are 3.91 m/s and 32.5 cm respectively?
Here $f = v/\lambda = (3.91\,\text{m/s})/(0.325\,\text{m}) = 12.0\,\text{Hz}$.

Exr 5.1.23 What is the frequency of a sound wave whose speed and wavelength are 343 m/s and 20.3 cm respectively?
Here $f = v/\lambda = (343\,\text{m/s})/(0.203\,\text{m}) = 1690\,\text{Hz}$.

Exr 5.1.24 What is the frequency of an ultrasonic wave traveling through bone whose speed and wavelength are 3290 m/s and 1.94 mm respectively?
$f = v/\lambda = (3290\ \text{m/s})/(0.00194\ \text{m}) = 1.70 \times 10^{+6}\ \text{Hz} = 1.70\,\text{MHz}$.

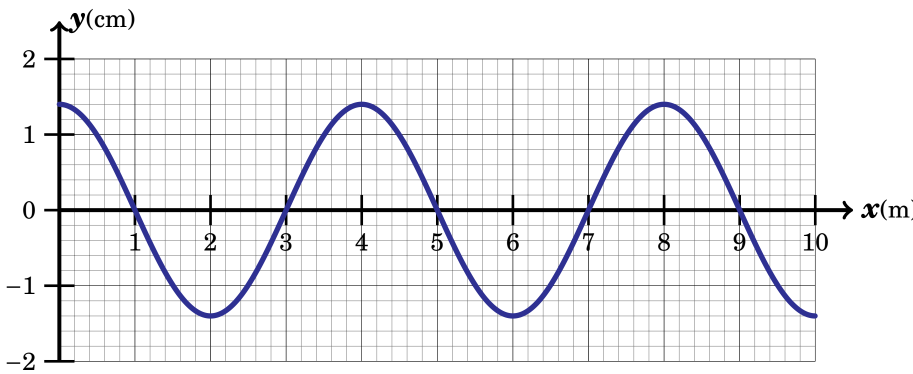
5.1.3 Graphs of Waves

Determining the Amplitude, Wavelength, and other parameters, of Pulses and Periodic Waves from Graphs.

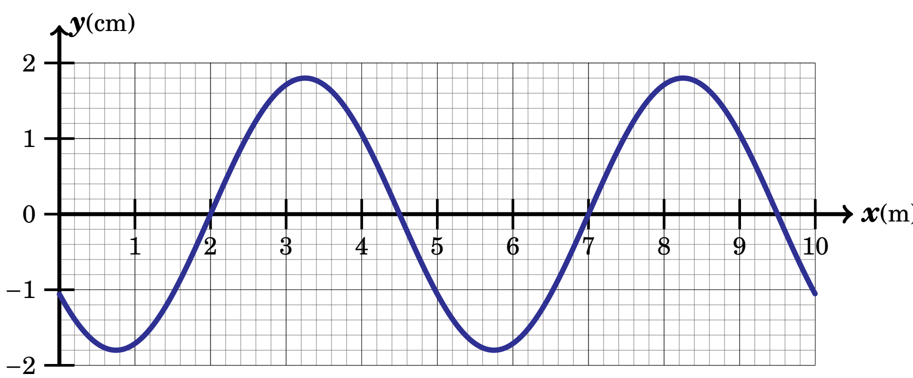
Terminology: A *pulse* is a wave that does not repeat. A pulse travels at the wavespeed of the medium, but does not have a wavelength or frequency because it does not repeat across space or across time.

Displacement versus Position

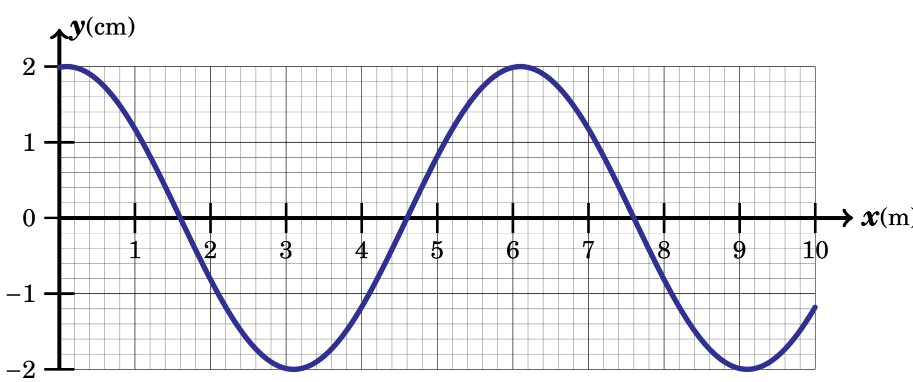
Exr 5.1.25 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 1.4\,\text{cm}$ and $\lambda = 4.0\,\text{m}$.



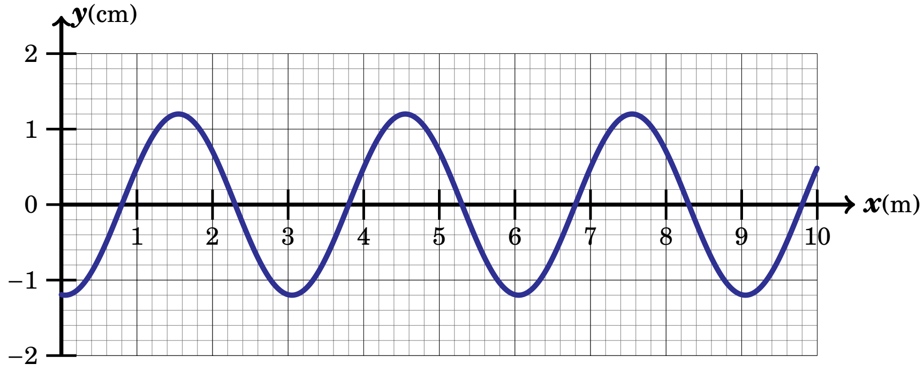
Exr 5.1.26 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 1.8\,\text{cm}$ and $\lambda = 5.0\,\text{m}$.



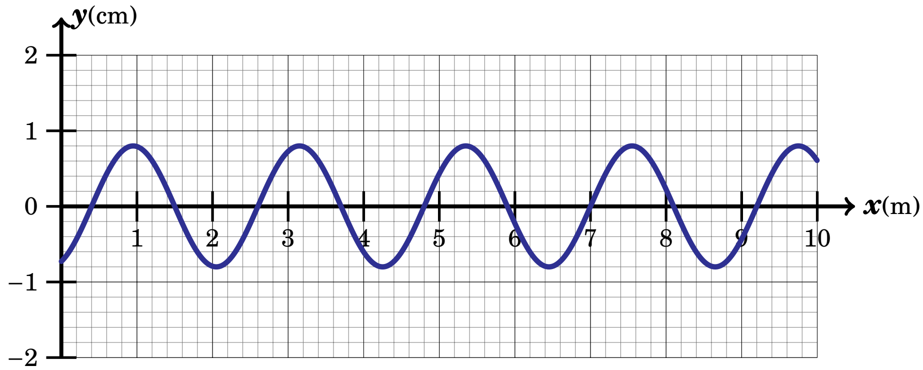
Exr 5.1.27 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 2.0\,\text{cm}$ and $\lambda = 6.0\,\text{m}$.



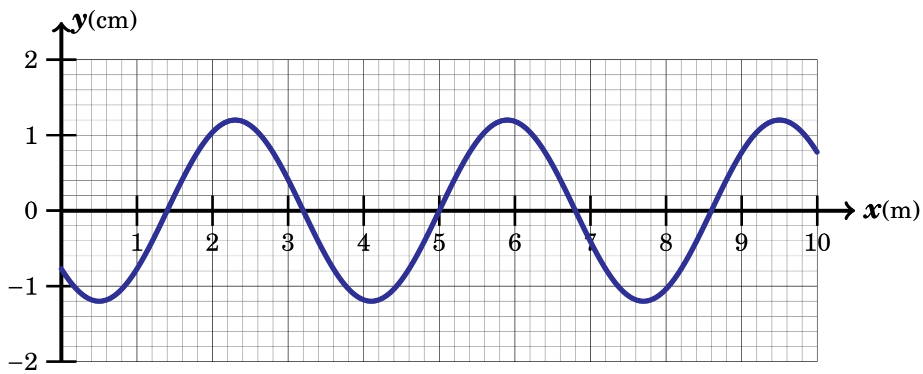
Exr 5.1.28 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 1.2\,\text{cm}$ and $\lambda = 3.0\,\text{m}$.



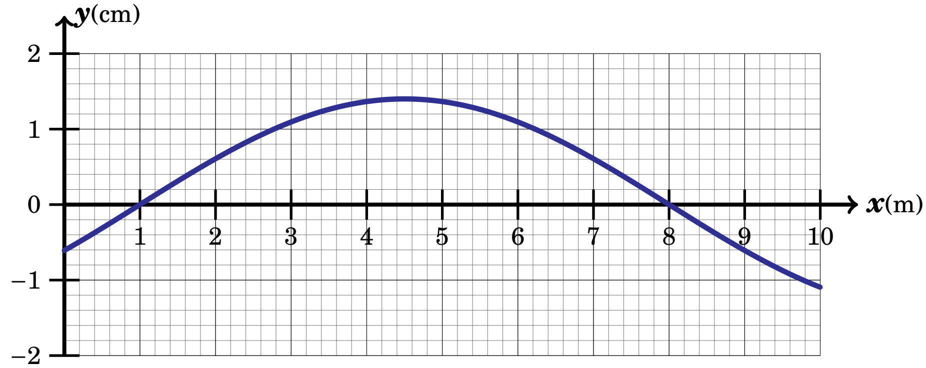
Exr 5.1.29 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 0.8\,\text{cm}$ and $\lambda = 2.2\,\text{m}$.



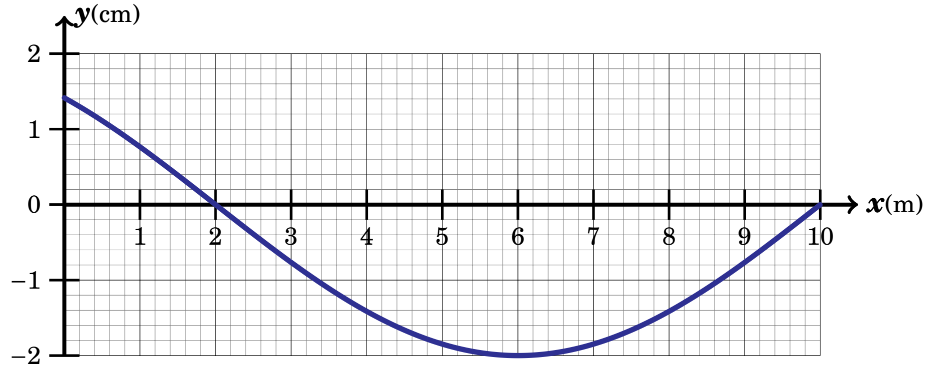
Exr 5.1.30 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 1.2\,\text{cm}$ and $\lambda = 3.6\,\text{m}$.



Exr 5.1.31 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 1.4\,\mathrm{cm}$ and $\lambda = 14.0\,\mathrm{m}$.

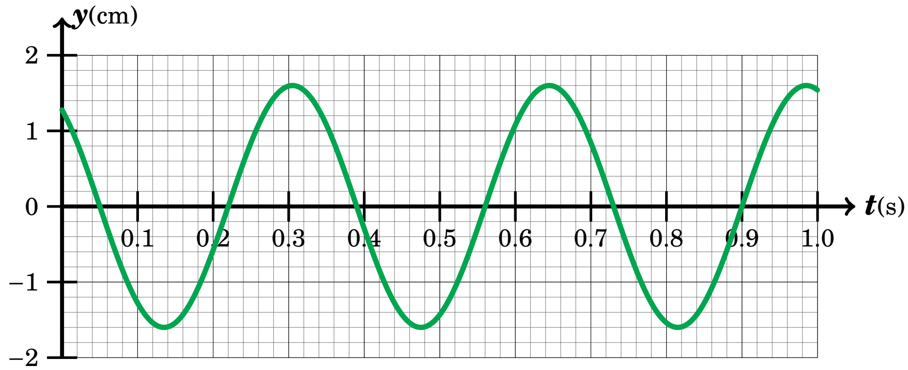


Exr 5.1.32 What are the amplitude and wavelength of the periodic wave graphed below?
Answer: $A = 2.0\,\mathrm{cm}$ and $\lambda = 16.0\,\mathrm{m}$.

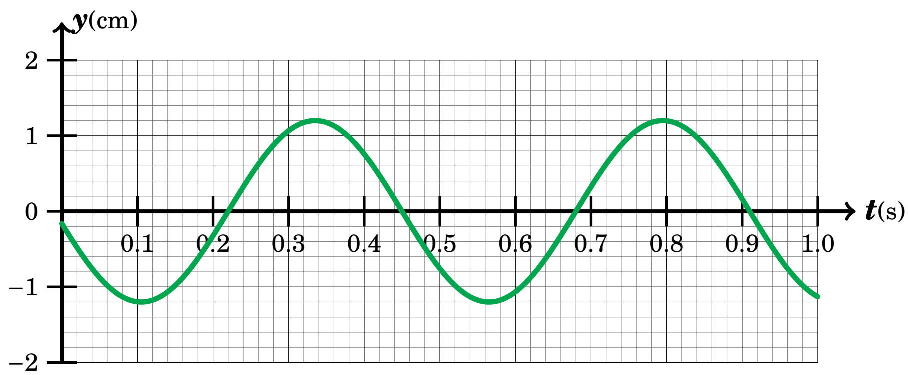


Displacement versus Time

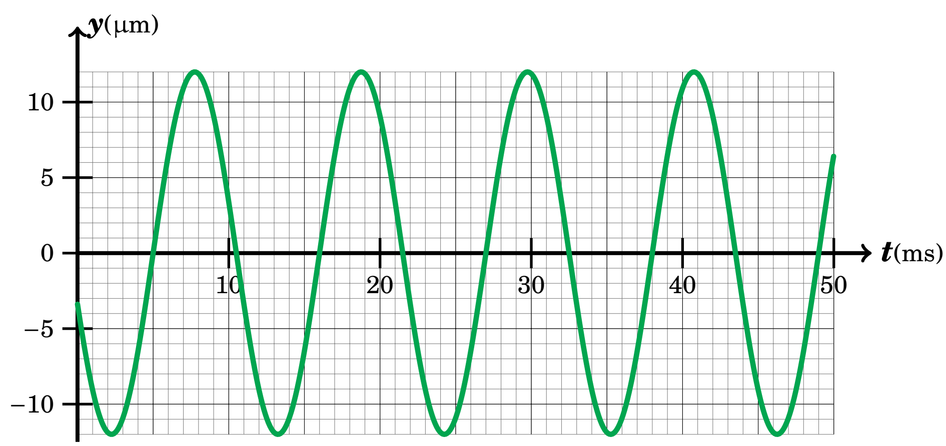
Exr 5.1.33 A periodic wave is traveling across a stretched string. Graphed below is the displacement as a function of *time* of a single piece of that string. What are the amplitude, period, and frequency of this wave?
Answer: $A = 1.6\,\mathrm{cm}$, $T = 0.34\,\mathrm{s}$, and $f = 2.94\,\mathrm{Hz}$.



Exr 5.1.34 A periodic wave is traveling across a stretched string. Graphed below is the displacement as a function of time of a single piece of that string. What are the amplitude, period, and frequency of this wave?
Answer: $A = 1.2\,\mathrm{cm}$, $T = 0.46\,\mathrm{s}$, and $f = 2.17\,\mathrm{Hz}$.

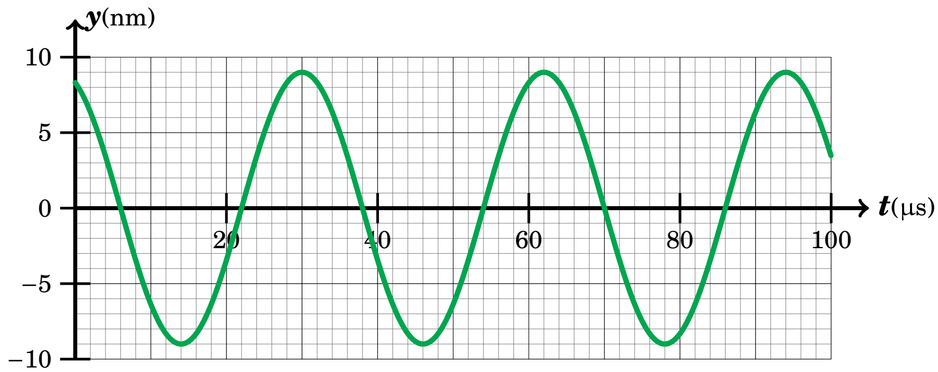


Exr 5.1.35 A periodic wave is traveling across a metal rod. Graphed below is the displacement as a function of time of a single piece of that object. What are the amplitude, period, and frequency of this wave?
Answer: $A = 12.0\,\mu\mathrm{m}$, $T = 11.0\,\mathrm{ms}$, and $f = 90.9\,\mathrm{Hz}$.



Exr 5.1.36 A periodic sound wave is traveling through water. Graphed below is the displacement as a function of time of a single portion of the medium. What are the amplitude, period, and frequency of this wave?

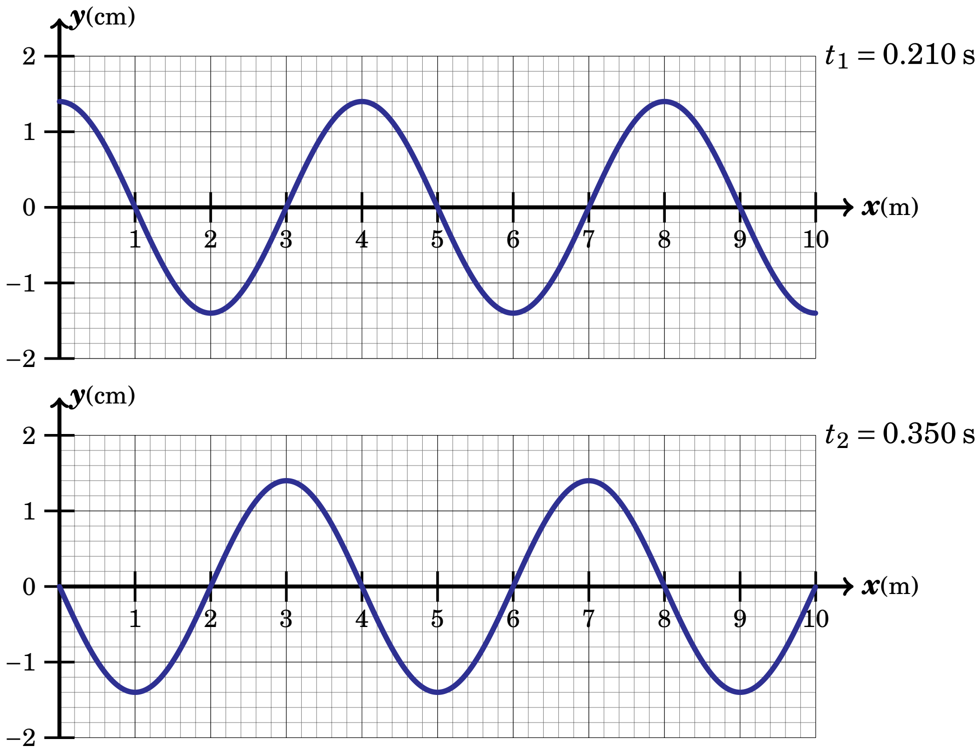
Answer: $A = 9.0\,\text{nm}$, $T = 32.0\,\mu\text{s}$, and $f = 31.3\,\text{kHz}$.



Pairs of Graphs

PROBLEM 5.1.01: Below are two photographs of a traveling periodic wave, taken at two different times. (The time between these photographs is less than half the period of oscillation of the source.) **(a)** What direction is the wave traveling, and at what speed? **(b)** What are the wavelength and frequency of this wave?

Answer: (a) Towards the left at $v = 7.14\,\mathrm{m/s}$. **(b)** $\lambda = 4.0\,\mathrm{m}$ and $f = 1.79\,\mathrm{Hz}$.



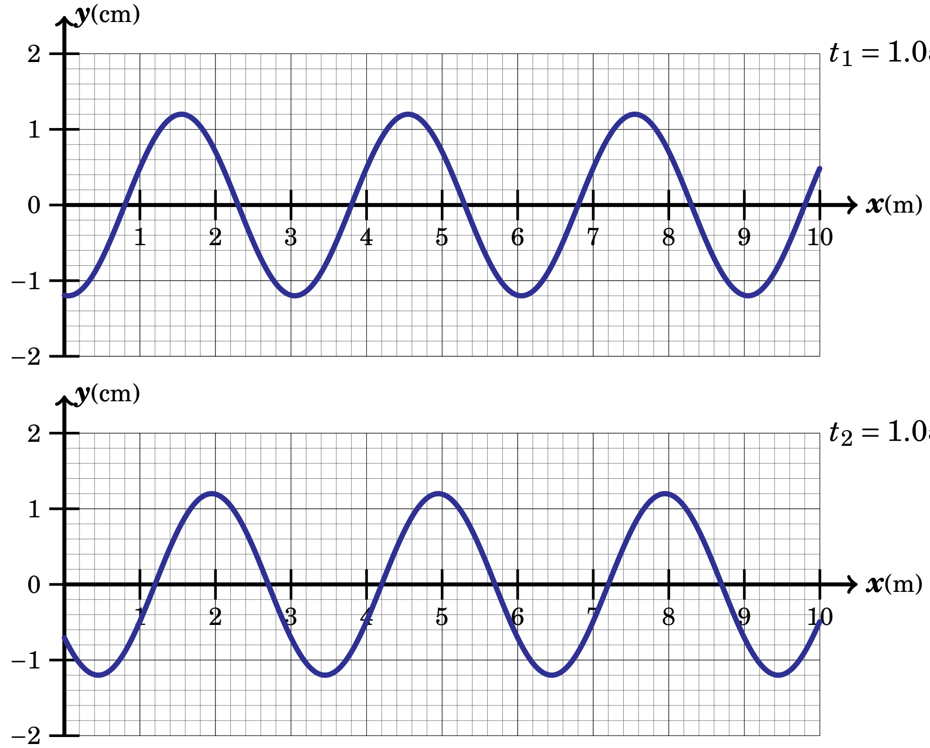
(a) To find the speed of the wave we can track the motion of a single feature of the wave's shape. For example, looking at the photograph of the wave at $t_1 = 0.210\,\mathrm{s}$ we can see where it see where it crosses the x-axis at $x_1 = 3.0\,\mathrm{m}$, sloping upwards. In the later photograph at $t_2 = 0.350\,\mathrm{s}$ we can see that that feature has moved to $x_2 = 2.0\,\mathrm{m}$. Consequently we find that the wave has moved $\Delta x = x_2 - x_1 = -1.0\,\mathrm{m}$ (one metre towards the left).

That displacement happens over a time interval $\Delta t = t_2 - t_1 = 0.140\,\mathrm{s}$. The velocity of the wave is thus $v_x = \Delta x/\Delta t = (-1.0\,\mathrm{m})/(0.140\,\mathrm{s}) = -7.14\,\mathrm{m/s}$. The wavespeed is $v = 7.14\,\mathrm{m/s}$, and the wave is traveling towards the left.

(b) From the second photograph it is easy to see that he wavelength is $\lambda = 4.0\,\mathrm{m}$. Due to the fundamental relation of periodic waves ($\lambda = v/f$) we get that $f = v/\lambda = (7.14\,\mathrm{m/s})/(4.0\,\mathrm{m}) = 1.79\,\mathrm{Hz}$.

PROBLEM 5.1.02: Below are two frames from a video of a traveling periodic wave. (The time between these photographs is less than half the period of oscillation of the source.) **(a)** What direction is the wave traveling, and at what speed? **(b)** What are the wavelength and frequency of this wave?

Answer: (a) Towards the right at $v = 200\,\text{m/s}$. **(b)** $\lambda = 3.0\,\text{m}$ and $f = 67\,\text{Hz}$.

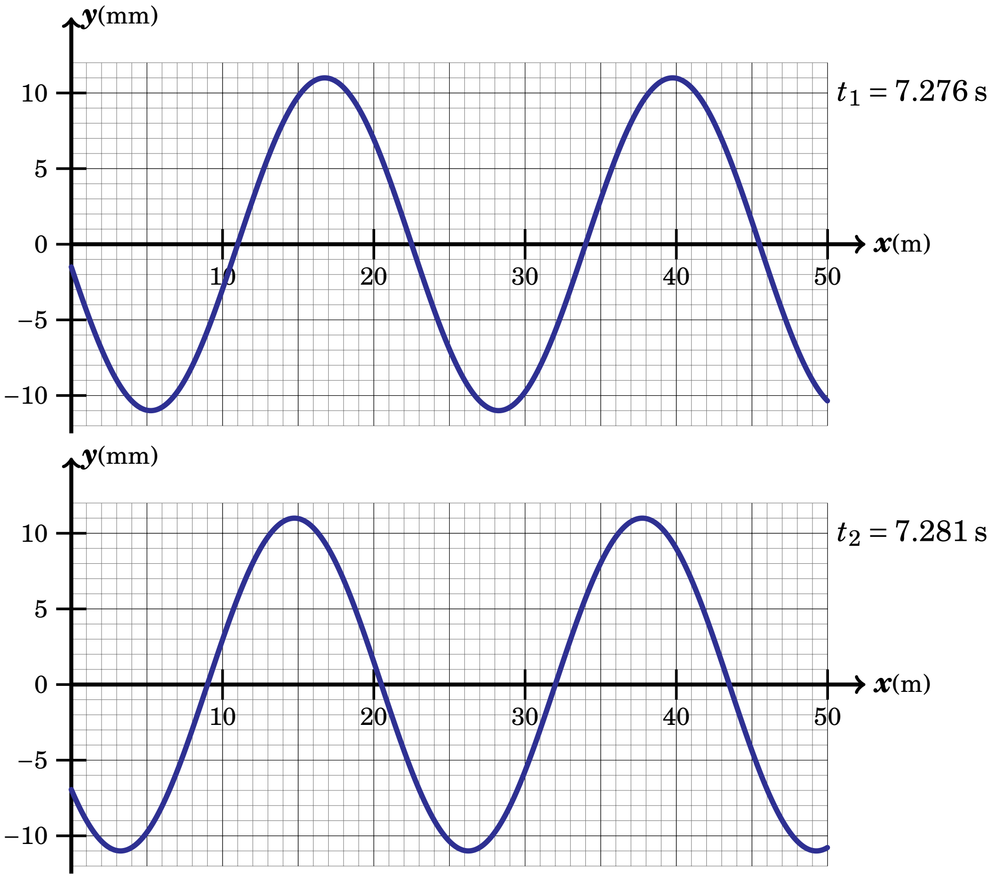


(a) Looking at where the displacement of the wave crosses the x-axis we can see that the wave has moved $\Delta x = +0.4\,\text{m}$ over a time interval $\Delta t = 0.002\,\text{s}$. The velocity of the wave is thus $v_x = \Delta x/\Delta t = (+0.4\text{ m})/(0.002\text{ s}) = +200\text{ m/s}$. The wavespeed is $v = 200\,\text{m/s}$, and the wave is traveling towards the right.

(b) The wavelength is $\lambda = 3.0\,\text{m}$. Due to the fundamental relation of periodic waves ($\lambda = v/f$) we get that $f = v/\lambda = (200\,\text{m/s})/(3.0\,\text{m}) = 67\,\text{Hz}$.

PROBLEM 5.1.03: Below are two frames from a video of a periodic wave traveling along a thin metal rod. (The time between these photographs is less than half the period of oscillation of the source.) **(a)** What direction is the wave traveling, and at what speed? **(b)** What are the wavelength and frequency of this wave?

Answer: (a) Towards the left at $v = 400\,\mathrm{m/s}$. **(b)** $\lambda = 23.0\,\mathrm{m}$ and $f = 17.4\,\mathrm{Hz}$.



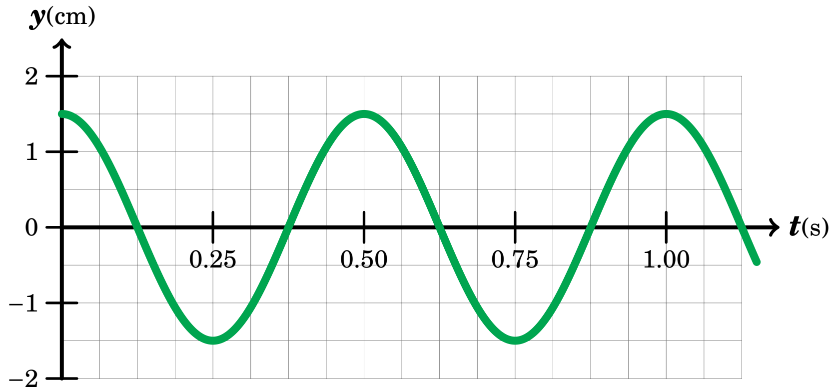
(a) Looking at where the displacement of the wave crosses the x-axis we can see that the wave has moved $\Delta x = -2.0\,\mathrm{m}$ over a time interval $\Delta t = 0.005\,\mathrm{s}$. The velocity of the wave is thus $v_x = \Delta x/\Delta t = (-2.0\,\mathrm{m})/(0.005\,\mathrm{s}) = -400\,\mathrm{m/s}$. The wavespeed is $v = 400\,\mathrm{m/s}$, and the wave is traveling towards the left.

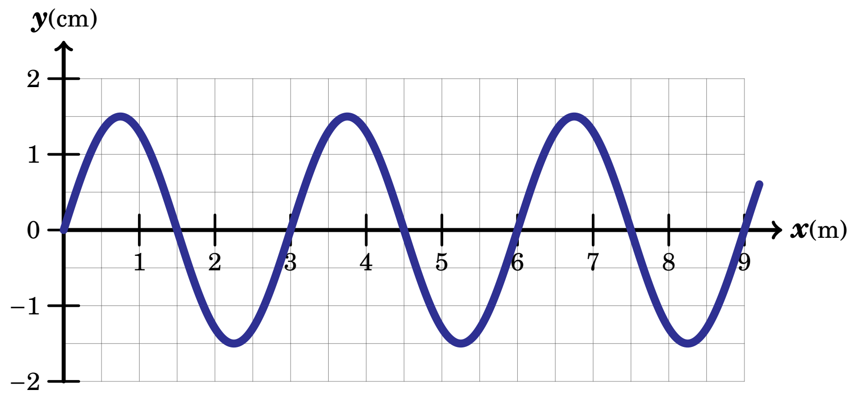
(b) The wavelength is $\lambda = 23.0\,\mathrm{m}$. Due to the fundamental relation of periodic waves ($\lambda = v/f$) we get that $f = v/\lambda = (400\,\mathrm{m/s})/(23.0\,\mathrm{m}) = 17.4\,\mathrm{Hz}$.

PROBLEM 5.1.04: The graphs below show measurements made of a traveling wave on a string. The first graph is the displacement (y) as a function of time (t) for a small piece of the string. The second graph is the displacement (y) as a function of position (x) along the length, taken at $t = 0\,\text{s}$.

Find **(a)** the amplitude, **(b)** the wavelength, **(c)** the period, **(d)** the frequency, and **(e)** the wavespeed of the wave.

Answers: (a) $A = 1.5\,\text{cm}$; **(b)** $\lambda = 3.0\,\text{m}$; **(c)** $T = 0.500\,\text{s}$; **(d)** $f = 2.00\,\text{Hz}$; and **(e)** $v = 6.0\,\text{m/s}$.





(a) The amplitude of the wave is found from the graph just by looking at it! The amplitude is the height of any of the peaks in either graph: $A = 1.5\,\text{cm}$.

(b) The wavelength is the distance between repetitions across space of the wave's shape. The graph of displacement (y) as a function of position (x) crosses the horizontal axis, sloping upwards, at $x = 3.0\,\text{m}$. Tracing along the contour of the string the next place where the string crosses the axis sloping upwards is at $x = 6.0\,\text{m}$. After that the shape of the wave repeats. Thus $\lambda = 6.0\,\text{m} - 3.0\,\text{m} = 3.0\,\text{m}$.

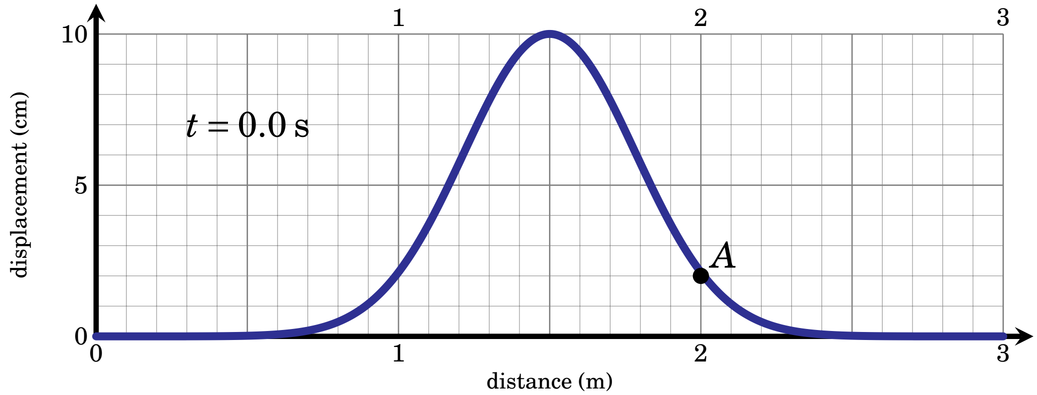
(c) The graph of displacement (y) as a function of time (t) crosses the horizontal axis, sloping upwards, at $t = 0.375\,\text{s}$. Tracing along the contour of the graph the next place where the line crosses the axis sloping upwards is at $t = 0.875\,\text{s}$. After that the oscillation repeats. Thus $T = 0.875\,\text{s} - 0.375\,\text{s} = 0.500\,\text{s}$.

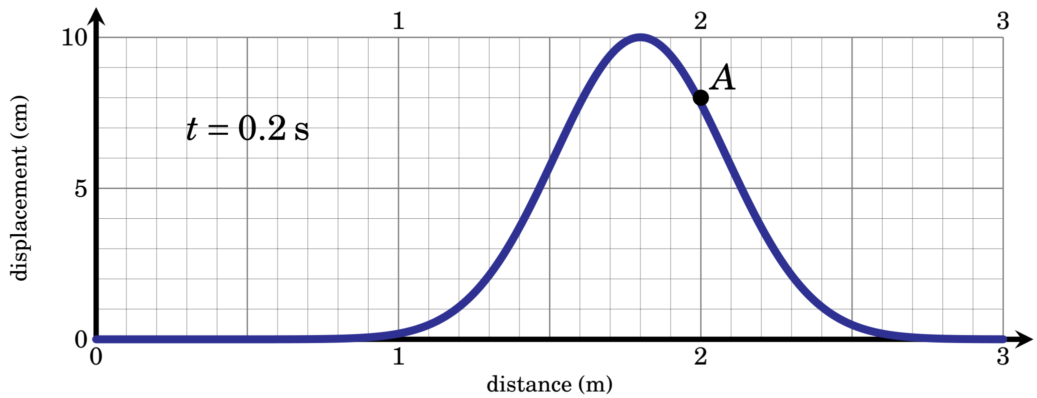
(d) By definition $f = 1/T = 1/(0.500\,\text{s}) = 2.00\,\text{Hz}$.

(e) From the fundamental relation for periodic waves ($\lambda = v/f$) we have $v = \lambda f = (3.0\,\text{m})(2.00\,\text{Hz}) = 6.0\,\text{m/s}$.

5.1.4 Problems

PROBLEM 5.1.05: The figures below show a pulse on a string at two times: $t = 0$ s and $t = 0.2$ s. Since this is a pulse, not a continuous periodic wave, so there is neither a "wavelength" nor a "period" of oscillation. (Note carefully that the horizontal and vertical scales are different.) **(a)** What is the speed of the pulse? **(b)** What is the vertical speed of the piece of string labeled "A" during this interval? **(c)** What is the position of the peak of the pulse at $t = 3.0$ s? **(d)** At what time will the pulse arrive at the $x = 4.0$ m position?
Answers: (a) $v = 1.5$ m/s; **(b)** $v = 0.30$ m/s; **(c)** $x = 6.0$ m; and **(d)** $t = 1.67$ s.





(a) There is no repetition in the shape of this wave – it is not a periodic wave. Thus we can not use $v = \lambda f$ to find the speed. The speed of the pulse will be determined by $v = \Delta x/\Delta t$, where Δx will be the distance traveled by a specific feature on the pulse – in this case, the top of the pulse.

At time $t = 0.0$ s the top of the pulse is at $x = 1.5$ m. At time $t = 0.2$ s the top of the pulse is at $x = 1.8$ m. Thus

$$v = \frac{\Delta x}{\Delta t} = \frac{(1.8\,\text{m} - 1.5\,\text{m})}{0.2\,\text{s}} = +1.5\,\text{m/s} \tag{5.6}$$

(b) The speed of the piece of the string named "A" will be given by $v = \Delta y/\Delta t$. From the graphs we find that at $t = 0.0$ s the piece "A" is at $y = 2.0$ cm, and that at $t = 0.2$ s it is at $y = 8.0$ cm. Thus

$$v = \frac{\Delta y}{\Delta t} = \frac{(0.080\,\text{m} - 0.020\,\text{m})}{0.2\,\text{s}} = +0.30\,\text{m/s} \tag{5.7}$$

Note carefully that this speed (the vertical motion of a physical piece of the string) is *independent* of the speed at which the wave travels.

(c) The position of the peak of the pulse will vary with time as $x_\text{f} = x_\text{i} + v\,\Delta t$. (Note: This equation, which originated from the study of the motion of objects [*kinematics*], applies here, even though the wave itself is not an object. In the context of this solitary pulse, this equation describes the location of the pulse – which is transporting energy from one place to another – as a function of time.) From the given data and the results of part (b), we obtain

$$x_\text{f} = x_\text{i} + v\,\Delta t \tag{5.8}$$
$$= (+1.5\,\text{m}) + (+1.5\,\text{m/s})(3.0\,\text{s}) \tag{5.9}$$
$$= 6.0\,\text{m} \tag{5.10}$$

(d) Solving the kinematic equation for the unknown time

$$x_\text{f} = x_\text{i} + v\,\Delta t \tag{5.11}$$

$$\Delta t = (x_\text{f} - x_\text{i})/v \tag{5.12}$$

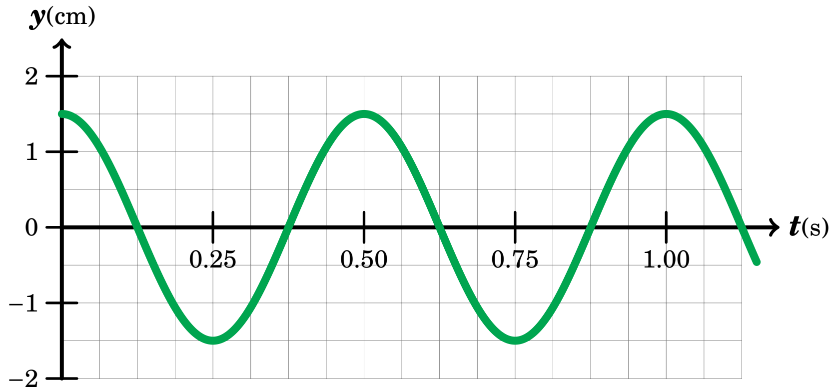
$$= (4.0\text{ m} - 1.5\text{ m})/(1.5\text{ m/s}) \tag{5.13}$$

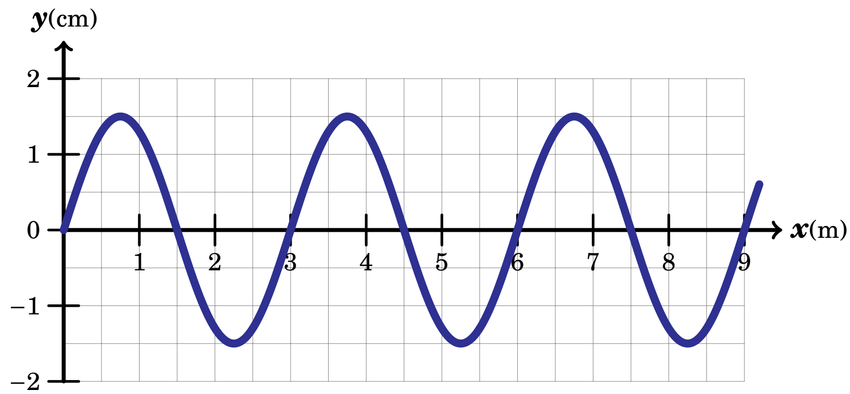
$$= 1.67\text{ s} \tag{5.14}$$

PROBLEM 5.1.06: The graphs below show measurements made of a traveling wave on a string. The first graph is the displacement (y) as a function of time (t) for a small piece of the string. The second graph is the displacement (y) as a function of position (x) along the length, taken at $t = 0$ s.

Find **(a)** the amplitude, **(b)** the wavelength, **(c)** the period, **(d)** the frequency, and **(e)** the wavespeed of the wave.

Answers: (a) $A = 1.5$ cm; **(b)** $\lambda = 3.0$ m; **(c)** $T = 0.500$ s; **(d)** $f = 2.00$ Hz; and **(e)** $v = 6.0$ m/s.





(a) The amplitude of the wave is found from the graph just by looking at it! The amplitude is the height of any of the peaks in either graph: $A = 1.5$ cm.

(b) The wavelength is the distance between repetitions across space of the wave's shape. The graph of displacement (y) as a function of position (x) crosses the horizontal axis, sloping upwards, at $x = 3.0$ m. Tracing along the contour of the string the next place where the string crosses the axis sloping upwards is at $x = 6.0$ m. After that the shape of the wave repeats. Thus $\lambda = 6.0\,\text{m} - 3.0\,\text{m} = 3.0\,\text{m}$.

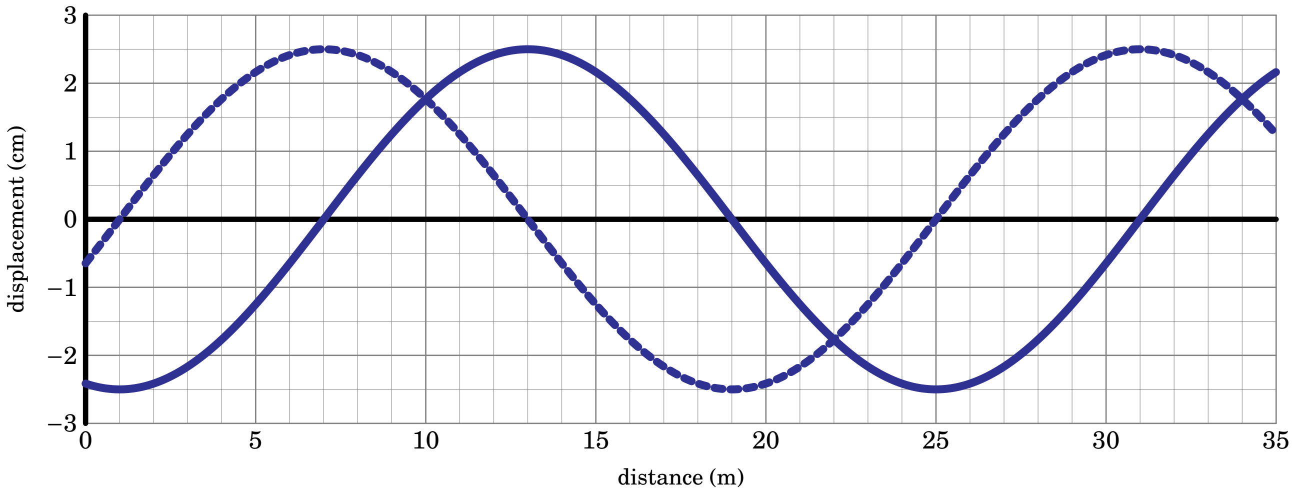
(c) The graph of displacement (y) as a function of time (t) crosses the horizontal axis, sloping upwards, at $t = 0.375$ s. Tracing along the contour of the graph the next place where the line crosses the axis sloping upwards is at $t = 0.875$ s. After that the oscillation repeats. Thus $T = 0.875\,\text{s} - 0.375\,\text{s} = 0.500\,\text{s}$.

(d) By definition $f = 1/T = 1/(0.500\,\text{s}) = 2.00\,\text{Hz}$.

(s) From the fundamental relation for periodic waves ($\lambda = v/f$) we have $v = \lambda f = (3.0\,\text{m})(2.00\,\text{Hz}) = 6.0\,\text{m/s}$.

PROBLEM 5.1.07: The graph below shows a sine wave traveling to the right on a string. The dashed line is the shape of the string at time $t = 0$ s, and the solid curve is the shape of the string at time $t = 0.12$ s. (Note carefully that the horizontal and vertical scales are different.) Find **(a)** the amplitude, **(b)** the wavelength, **(c)** the speed, **(d)** the frequency, and **(e)** the period of the wave.

Answers: **(a)** $A = 2.5$ cm; **(b)** $\lambda = 24$ m; **(c)** $v = 50$ m/s; **(d)** $f = 2.08$ Hz; and **(e)** $T = 0.48$ s.



(a) The amplitude of the wave is found from the graph just by looking at it! On the graph of the wave at $t = 0$ s there is a peak at $x = 7$ m and $y = 2.5$ cm. The amplitude is the height of this peak: $A = 2.5$ cm.

(b) The wavelength is the distance between repetitions of the wave's shape. The graph of the wave at $t = 0$ s (the dashed curve) crosses the x-axis at $x = 1$ m sloping upwards. Tracing along the contour of the string at that time the next place where the string slopes upwards is at $x = 25$ m. After that the shape of the wave repeats. Thus $\lambda = 25\text{ m} - 1\text{ m} = 24$ m.

(c) At $t = 0$ s (the dashed curve) the string slopes upwards crossing the x-axis at $x = 1$ m. At $t = 0.12$ s (the solid curve) that same portion of the wave's shape is at $x = 7$ m. Thus $v = \Delta x/\Delta t = (7\text{ m} - 1\text{ m})/(0.12\text{ s}) = 50$ m/s.

(d) From the fundamental relation for periodic waves ($\lambda = v/f$) we have $f = v/\lambda = (50\text{ m/s})/(24\text{ m}) = 2.08$ Hz.

(e) By definition $T = 1/f = 1/(2.08\text{ Hz}) = 0.48$ s.

5.2 Sound

Traveling waves in air, water and people.

5.2.1 Intensity, Power, and Area

Intensity measures the way in which the power transferred by a wave is distributed across an area:

$$I = P/A \tag{5.15}$$

Intensity is typically measured in watts per square metre W/m^2, but in some circumstances (when the areas involved are human-sized) it is measured in watts per square centimetre W/cm^2. It is important to remember that while lengths convert as $1\,m = 100\,cm$, areas convert as $1\,m^2 = (100\,cm)^2 = 10\,000\,cm^2$, so that $1\,cm^2 = 10^{-4}\,m^2$. In all cases pay careful attention to the units of the lengths and the units of the area(s) involved.

When a wave spreads out in three dimensions, the wavefronts are spherical. The surface area of a sphere is given by

$$A_{\text{sphere}} = 4\pi r^2 \tag{5.16}$$

(Yes, this differs from the formula for the area of a flat circle by a factor of four, so be careful not to mix them up.) Because of this, the intensity a distance r from a source of power P (producing either sound or light) is

$$I = \frac{P}{4\pi r^2} \tag{5.17}$$

This is known as the *inverse-square law* for intensity.

Exr 5.2.01 Find the intensity (measured in W/m^2) of a wave that delivers 16.0 W to a rectangular area that measures 23 cm by 71 cm.
Since we are asked for the intensity to be measured per square metre, we should convert distances to metres first: $I = P/A = 16.0\,W/(0.23\,m \times 0.71\,m) = 98\,W/m^2$.

Exr 5.2.02 Find the intensity (measured in W/m^2) of a wave that delivers 72.5 W to the surface of a door that measures 1.05 m by 1.82 m.
$I = P/A = 72.5\,W/(1.05\,m \times 1.82\,m) = 37.9\,W/m^2$.

Exr 5.2.03 Find the intensity (measured in W/m^2) of a wave that delivers 60.0 W to a rectangular area that measures 2.50 m by 3.75 m.
$I = P/A = 60.0\,W/(2.50\,m \times 3.75\,m) = 6.4\,W/m^2$.

Exr 5.2.04 Find the intensity (measured in W/m^2) on a soccer field (measuring 73 m by 110 m) illuminated by 7.50 kW of lighting.
$I = P/A = 7500\,W/(73\,m \times 110\,m) = 0.934\,W/m^2$.

Exr 5.2.05 Find the intensity (measured in W/cm^2) of a wave that delivers 90.4 W to a rectangular area that measures 7.5 cm by 11.0 cm.
$I = P/A = 90.4\,W/(7.5\,cm \times 11.0\,cm) = 1.1\,W/cm^2$.

Exr 5.2.06 Find the intensity (measured in W/cm^2) of a wave that delivers 72.5 W to the surface of a door that measures 1.05 m by 1.82 m.
Since we are asked for the intensity to be measured per square centimetre, we should convert distances to centimetres first: $I = P/A = 72.5\,W/(105\,cm \times 182\,cm) = 0.00379\,W/cm^2 = 3.79\,mW/cm^2$.

Exr 5.2.07 Find the intensity (measured in W/cm^2) of light that delivers 15.0 W onto a piece of paper (measuring 210 mm by 297 mm).
Since we are asked for the intensity to be measured per square centimetre, we should convert distances to centimetres first: $I = P/A = 15.0\,W/(21.0\,cm \times 29.7\,cm) = 0.0241\,W/cm^2$.

Exr 5.2.08 Find the intensity (measured in W/cm^2) of a computer monitor that emits 0.500 W of light energy. (measuring 274 mm by 487 mm).
Since we are asked for the intensity to be measured per square centimetre, we should convert distances to centimetres first: $I = P/A = 0.500\,W/(27.4\,cm \times 48.7\,cm) = 0.000375\,W/cm^2 = 0.375\,mW/cm^2$.

Exr 5.2.09 Find the intensity (measured in W/m^2) of a wave that delivers 13.7 W to a circular area 5.00 m in diameter.
Always remember that when given the diameter of a circle we will need the *radius* to find the area $A = \pi r^2$. $I = P/A = 13.7\,W/\left(\pi(\frac{1}{2} \times 5.00\,m)^2\right) = 0.698\,W/m^2$.

Exr 5.2.10 Find the intensity (measured in W/m^2) of a wave that delivers 88 mW to a circular area 75.2 cm in diameter.
Always remember that when given the diameter of a circle we will need the *radius* to find the area $A = \pi r^2$. $I = P/A = 0.088\,W/\left(\pi(\frac{1}{2} \times 0.752\,m)^2\right) = 0.20\,W/m^2$.

Exr 5.2.11 Find the intensity (measured in W/cm^2) of a wave that delivers 0.42 W to a circular area 6.2 cm in diameter.
Always remember that when given the diameter of a cir-

cle we will need the *radius* to find the area $A = \pi r^2$. $I = P/A = 0.42\,\mathrm{W}/\left(\pi(\frac{1}{2} \times 6.2\,\mathrm{cm})^2\right) = 0.014\,\mathrm{W/cm^2}$.

Exr 5.2.12 Find the intensity (measured in $\mathrm{W/cm^2}$) of a wave that delivers 72 W to a circular area 3.20 m in diameter.
Always remember that when given the diameter of a circle we will need the *radius* to find the area $A = \pi r^2$. $I = P/A = 72\,\mathrm{W}/\left(\pi(\frac{1}{2} \times 320\,\mathrm{cm})^2\right) = 0.00090\,\mathrm{W/cm^2} = 0.90\,\mathrm{mW/cm^2}$.

Exr 5.2.13 Find the intensity of a wave measured 5.00 m from a 4.20 W source.
We are given that the surface area of a sphere is $A = 4\pi r^2$. At the distance $r = 5.00\,\mathrm{m}$ from the source the wavefront is a sphere with radius r. Thus the intensity is $I = P/A = 4.20\,\mathrm{W}/\left(4\pi(5.00\,\mathrm{m})^2\right) = 0.0134\,\mathrm{W/m^2} = 13.4\,\mathrm{mW/m^2}$.

Exr 5.2.14 Find the intensity of a wave measured 5.42 m from a 0.37 W source.
$I = P/A = 0.37\,\mathrm{W}/\left(4\pi(5.42\,\mathrm{m})^2\right) = 0.0010\,\mathrm{W/m^2} = 1.0\,\mathrm{mW/m^2}$.

Exr 5.2.15 Find the intensity (measured in units of $\mathrm{W/cm^2}$) of a wave measured 6.2 cm from a 17.4 W source.
$I = P/A = 17.4\,\mathrm{W}/\left(4\pi(6.2\,\mathrm{cm})^2\right) = 0.036\,\mathrm{W/cm^2}$.

Exr 5.2.16 Find the intensity (measured in units of $\mathrm{W/cm^2}$) of a wave measured 2.56 m from a 0.909 W source.
$I = P/A = 0.909\,\mathrm{W}/\left(4\pi(256\,\mathrm{cm})^2\right) = 1.10 \times 10^{-6}\,\mathrm{W/cm^2}$. Since this quantity is so small, it would be better to measure it in units of *micro*-watts per square centimetre: $I = 1.10\,\mu\mathrm{W/cm^2}$

5.2.2 Intensity, Energy, and Time

Since power relates to energy and time, we can relate intensity to the amount of energy that was transferred:

$$P = I \times A \tag{5.18}$$

$$\Delta E = P \times \Delta t = I \times A \times \Delta t \tag{5.19}$$

These relations can also allow us to solve for the time required to deliver a required amount of energy:

$$\Delta t = \Delta E \big/ (I \times A) \tag{5.20}$$

If the source produces a wave whose amplitude is modulated, then the time required will be longer than that given by the equation above. The most common is when the source operates on a *duty cycle*, alternating between being on then off repeatedly. If $\delta \leq 1$ is the fraction of time that the source is "on", then $\Delta t_{\text{on}} = \delta \times \Delta t_{\text{total}}$, so that

$$\Delta t_{\text{on}} = \Delta E \big/ (I \times A) \tag{5.21}$$

$$\Delta t_{\text{total}} = \Delta t_{\text{on}} \big/ \delta \tag{5.22}$$

where I is the intensity of the wave when the source is "on".

Where necessary recall that $1\,\text{Cal} = 4\,184.\,\text{J}$ is the nutritional calorie, and that $1\,\text{kWh} = 3.6\,\text{MJ}$ is the usual unit of hydro-electrical energy (where each of these conversions are exact numbers).

Exr 5.2.17 Find the energy (measured in J) delivered by a wave of intensity $2.11\,\text{W/m}^2$ to a rectangular area that measures 1.24 m by 75 cm after 51 seconds.
Since the intensity is measured per square metre, the area must be measured in square metres, and we must convert the measurements of the area to metres. $\Delta E = I \times A \times \Delta t = 2.11\,\text{W/m}^2 \times (1.24\,\text{m} \times 0.75\,\text{m}) \times (51\,\text{s}) = 100\,\text{J}$.

Exr 5.2.18 Find the energy (measured in J) delivered by a wave of intensity $37.0\,\text{mW/m}^2$ to an area of $4.00\,\text{m}^2$ after one hour.
$\Delta E = I \times A \times \Delta t = 0.0370\,\text{W/m}^2 \times 4.00\,\text{m}^2 \times (60 \times 60\,\text{s}) = 533\,\text{J}$.

Exr 5.2.19 Find the energy (measured in kWh) delivered by a microwave oven (intensity $2.560\,\text{kW/m}^2$) to an area measuring 28 cm by 23 cm after 33 minutes.
$\Delta E = I \times A \times \Delta t = 2560\,\text{W/m}^2 \times (0.28\,\text{m} \times 0.23\,\text{m}) \times (33 \times 60\,\text{s}) = 3.26 \times 10^5\,\text{J} = 0.0907\,\text{kWh}$. (This calculation converted the power to watts so that we could do our time in seconds, but we could have kept it in kilowatts and used our time in hours to get kWh directly.)

Exr 5.2.20 Find the energy (measured in kWh) delivered by a wave of intensity $13.0\,\text{W/m}^2$ to an area of $15.0\,\text{m}^2$ after eight hours.
$\Delta E = I \times A \times \Delta t = 13.0\,\text{W/m}^2 \times 15.0\,\text{m}^2 \times (8 \times 60 \times 60\,\text{s}) = 5.62 \times 10^6\,\text{J} = 1.56\,\text{kWh}$.

Exr 5.2.21 Find the energy (measured in J) delivered by a wave of intensity $0.64\,\text{W/cm}^2$ to a rectangular area that measures 23 cm by 71 cm after 5 minutes.
Since the power is given in multiples of watts, the time must be measured in seconds: $\Delta E = I \times A \times \Delta t = 0.64\,\text{W/cm}^2 \times (23\,\text{cm} \times 71\,\text{cm}) \times (5 \times 60\,\text{s}) = 313\,536\,\text{J} = 0.314\,\text{MJ}$.

Exr 5.2.22 Find the energy (measured in J) delivered by a wave of intensity $0.053\,\text{W/cm}^2$ to a rectangular area that measures 17 cm by 20 cm after 3 minutes and 12 seconds.
Since the power is given in multiples of watts, the time must be measured in seconds: $\Delta E = I \times A \times \Delta t = 0.053\,\text{W/cm}^2 \times (17\,\text{cm} \times 20\,\text{cm}) \times (3 \times 60\,\text{s} + 12\,\text{s}) = 3\,460\,\text{J} = 3.5\,\text{kJ}$.

Exr 5.2.23 Find the energy (measured in Cal) delivered by a wave of intensity $12.0\,\text{W/cm}^2$ to an area of $200\,\text{cm}^2$ after seven and a half minutes.
Since the power is given in multiples of watts, the time must be measured in seconds: $\Delta E = I \times A \times \Delta t = 12.0\,\text{W/cm}^2 \times 200\,\text{cm}^2 \times (7.5 \times 60\,\text{s}) = 1.08 \times 10^6\,\text{J} = 258\,\text{Cal}$.

Exr 5.2.24 Find the energy (measured in Cal) delivered by a wave of intensity $9.50\,\text{W/cm}^2$ to a circular area of diameter 18.0 cm after fifteen minutes.
Since the power is given in multiples of watts, the time must be measured in seconds: $\Delta E = I \times A \times \Delta t = 9.50\,\text{W/cm}^2 \times \left(\pi(\tfrac{1}{2} \times 18.0\,\text{cm})^2\right) \times (15 \times 60\,\text{s}) = 2.18 \times 10^6\,\text{J} = 520\,\text{Cal}$.

Exr 5.2.25 Find the time required (measured in minutes and seconds) to deliver 513 kJ over a rectangular area measuring 20.6 cm by 19.4 cm using a source with intensity $32\,\text{W/cm}^2$.
$\Delta t = \Delta E \big/ (I \times A) = (513 \times 10^3\,\text{J}) \big/ (32\,\text{W/cm}^2 \times (20.6\,\text{cm} \times 19.4\,\text{cm})) = 40\,\text{s}$.

Exr 5.2.26 Find the time required (measured in minutes and seconds) to deliver 1.20 MJ over a square area measuring 0.50 m on each side using a source with intensity $6.4\,\text{W/cm}^2$.
$\Delta t = \Delta E \big/ (I \times A) = (1.20 \times 10^6\,\text{J}) \big/ (6.4\,\text{W/cm}^2 \times (50\,\text{cm})^2) = 75\,\text{s} = 1\,\text{min and } 15\,\text{s}$.

Exr 5.2.27 Find the time required (measured in minutes and seconds) to deliver 0.765 Cal over a rectangular area measuring 21 cm by 23 cm using a source with intensity 5.23 mW/cm^2.
$\Delta t = \Delta E/(I \times A)$
$= (0.765\,\text{Cal} \times 4\,184\,\text{J/Cal})/(0.00523\,\text{W/cm}^2 \times (21\,\text{cm} \times 23\,\text{cm})$
$= 1267\,\text{s} = 21\,\text{min and } 7\,\text{s}.$

Exr 5.2.28 Find the time required (measured in minutes and seconds) to deliver 82.0 Cal over a circular area of diameter 15.0 cm using a source with intensity 7.00 W/cm^2.
$\Delta t = \Delta E/(I \times A)$
$= (82.0\,\text{Cal} \times 4\,184\,\text{J/Cal})/(7.00\,\text{W/cm}^2 \times \left(\pi(\tfrac{1}{2} \times 15.0\,\text{cm})^2\right))$
$= 277\,\text{s} = 4\,\text{min and } 37\,\text{s}.$

Exr 5.2.29 Find the time required (measured in minutes and seconds) to deliver 513 kJ over a rectangular area measuring 20.6 cm by 19.4 cm using a source with intensity 32 W/cm^2 operating on a 20% duty cycle.
The time the source needs to be on is $\Delta t_{\text{on}} = \Delta E/(I \times A) = (513 \times 10^3\,\text{J})/(32\,\text{W/cm}^2 \times (20.6\,\text{cm} \times 19.4\,\text{cm})) = 40\,\text{s}$. Since this is operating on a 20% duty cycle, the fraction of time it is on is $\delta = 0.20$, so that it will take $\Delta t_{\text{total}} = (\Delta t_{\text{on}})/\delta = (40\,\text{s})/0.20 = 200\,\text{s} = 3\,\text{min and } 20\,\text{s}$ to deliver the required energy.

Exr 5.2.30 Find the time required (measured in minutes and seconds) to deliver 62.0 kJ over a circular area 8.7 cm in diameter using a source with intensity 17.3 W/cm^2 operating on a 10% duty cycle.
The time the source needs to be on is $\Delta t_{\text{on}} = \Delta E/(I \times A) = (62.0 \times 10^3\,\text{J})/(17.3\,\text{W/cm}^2 \times \left(\pi(\tfrac{1}{2} \cdot 8.7\,\text{cm})^2\right)) = 60.2\,\text{s}$. Since this is operating on a 10% duty cycle, the fraction of time it is on is $\delta = 0.10$, so that it will take $\Delta t_{\text{total}} = (\Delta t_{\text{on}})/\delta = (60.2\,\text{s})/0.10 = 602.\,\text{s} = 10\,\text{min and } 2\,\text{s}$ (10 minutes, essentially) to deliver the required energy.

5.2.3 Sound Level

The quantitative measure β that models subjective loudness is ***sound level***, defined by:

$$\beta = (10\,\mathrm{dB})\log\left(I/I_0\right) \tag{5.23}$$

The logarithm returns a number, and the factor 10 dB expresses the level as a multiple of the unit decibel (dB) that measures level. The argument of the logarithm is the ratio of the intensity I of the sound wave to I_0 the *reference intensity* which is defined to be the exact quantity

$$I_0 = 1\times 10^{-12}\,\mathrm{W/m^2} \tag{5.24}$$

which corresponds (roughly) to the quietest sound that an average person can hear.

In cases where the level is known and the intensity is asked for, the relation is

$$I = I_0 \times 10^{\beta/10\,\mathrm{dB}} \tag{5.25}$$

Take care with the exponents in these calculations! Ideally you will figure out how to get your calculator to express all quantities in *scientific notation*.

Exr 5.2.31 Find the sound level (in decibels) of a sound that has intensity $1\times10^{-6}\,\mathrm{W/m^2}$.
From the definition $\beta = (10\,\mathrm{dB})\log\left(I/I_0\right) = (10\,\mathrm{dB})\log\left(\frac{1\times10^{-6}\,\mathrm{W/m^2}}{1\times10^{-12}\,\mathrm{W/m^2}}\right) = (10\,\mathrm{dB})\log(10^{+6}) = (10\,\mathrm{dB})\times(+6) = 60\,\mathrm{dB}$.

Exr 5.2.32 Find the sound level (in decibels) of a sound that has intensity $7.5\times10^{-8}\,\mathrm{W/m^2}$.
From the definition $\beta = (10\,\mathrm{dB})\log\left(I/I_0\right) = (10\,\mathrm{dB})\log\left(\frac{7.5\times10^{-8}\,\mathrm{W/m^2}}{1\times10^{-12}\,\mathrm{W/m^2}}\right) = 49\,\mathrm{dB}$.

Exr 5.2.33 Find the intensity (measured in $\mu\mathrm{W/m^2}$) of a sound of level 65 dB.
$I = I_0\times10^{\beta/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{65\,\mathrm{dB}/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{+6.5} = 1\times10^{-5.5}\,\mathrm{W/m^2} = 3.16\times10^{-6}\,\mathrm{W/m^2} = 3.16\,\mu\mathrm{W/m^2}$.

Exr 5.2.34 Find the intensity (measured in $\mu\mathrm{W/m^2}$) of a sound of level 53 dB.
$I = I_0\times10^{\beta/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{53\,\mathrm{dB}/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{+5.3} = 0.20\,\mu\mathrm{W/m^2}$.

Problem 5.2.01: Find the time required (measured in minutes and seconds) to deliver 1.37 μJ over an area of $21.0\,\mathrm{cm^2}$ using a source with a sound level of 70 dB.

From the level we get the intensity: $I = I_0\times10^{\beta/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{70\,\mathrm{dB}/10\,\mathrm{dB}} = 1\times10^{-5}\,\mathrm{W/m^2}$.

From the intensity and area we get power (the rate at which energy is being delivered to the area): $P = I\times A = (1\times10^{-5}\,\mathrm{W/m^2})\times(0.00210\,\mathrm{m^2}) = 21.0\times10^{-9}\,\mathrm{W}$.

From the power and the target energy, we find the required time: $\Delta t = \Delta E/P = (1.37\times10^{-6}\,\mathrm{J})/(21.0\times10^{-9}\,\mathrm{W}) = 65.2\,\mathrm{s}$. Just a little more than one minute.

Problem 5.2.02: Find the energy delivered (measured in microjoules) over a circular area of diameter 14.7 cm using a source with a sound level of 80 dB after two and a half minutes.

From the level we get the intensity: $I = I_0\times10^{\beta/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{80\,\mathrm{dB}/10\,\mathrm{dB}} = 1\times10^{-4}\,\mathrm{W/m^2}$.

From the intensity and area we get power (the rate at which energy is being delivered to the area): $P = I\times A = (1\times10^{-4}\,\mathrm{W/m^2})\times\left(\pi(\tfrac{1}{2}\times0.147\,\mathrm{m})^2\right) = 1.697\times10^{-6}\,\mathrm{W}$.

From the power and the time, we find the delivered energy: $\Delta E = P\times\Delta t = (1.697\times10^{-6}\,\mathrm{W})\times(2.5\times60\,\mathrm{s}) = 255.\,\mu\mathrm{J}$.

Problem 5.2.03: Find the energy delivered (measured in microjoules) over a circular area of $0.60\,\mathrm{cm^2}$ using a source with a sound level of 80 dB after three minutes.

From the level we get the intensity: $I = I_0\times10^{\beta/10\,\mathrm{dB}} = (1\times10^{-12}\,\mathrm{W/m^2})\times10^{80\,\mathrm{dB}/10\,\mathrm{dB}} = 1\times10^{-4}\,\mathrm{W/m^2}$.

From the intensity and area we get power (the rate at which energy is being delivered to the area): $P = I\times A = (1\times10^{-4}\,\mathrm{W/m^2})\times(0.60\times10^{-4}\,\mathrm{m^2}) = 0.60\times10^{-8}\,\mathrm{W}$.

From the power and the time, we find the delivered energy: $\Delta E = P\times\Delta t = (0.60\times10^{-8}\,\mathrm{W})\times(3\times60\,\mathrm{s}) = 1.08\,\mu\mathrm{J}$.

Problem 5.2.04: Ultrasound waves can have very large intensities without posing a danger to human hearing since no portion of the inner ear responds to such high frequencies. If we have such a wave, of intensity $10^{+5}\,\mathrm{W/m^2}$, what would be:

(a) The sound level of the wave?

(b) The total energy delivered to $1\,\mathrm{cm}^2$ after one minute?

(a) The level is $\beta = (10\,\mathrm{dB})\log\left(\frac{10^{+5}\,\mathrm{W/m^2}}{10^{-12}\,\mathrm{W/m^2}}\right) = 170\,\mathrm{dB}$. (If this were an audible sound wave, it would instantly destroy a person's eardrums.)

(b) The total energy delivered is $\Delta E = I \times A \times \Delta t = (10^{+5}\,\mathrm{W/m^2})(1 \times 10^{-4}\,\mathrm{m^2})(60\,\mathrm{s}) = 600\,\mathrm{J}$. (This is enough to raise the temperature of $100\,\mathrm{g}$ of water by $1.4\,\mathrm{C°}$.)

5.2.4 Sound Attenuation

The energy content of sound waves decreases with distance traveled due to internal friction in the medium. This process is called ***attenuation***. Attenuation relates the distance traveled by the wave (Δx) to the change in sound level ($\Delta\beta$):

$$\alpha = -\Delta\beta/\Delta x \tag{5.26}$$

The constant α is the attenuation coefficient. It is a property of the medium that depends upon the frequency of the wave. In the context of sound waves traveling through air α is usually measured in units of dB/km. In the context of ultrasound traveling through water or the human body α is usually measured in units of dB/cm.

Calculating energy deposition from attenuation: A drop in intensity to a fraction $I_2/I_1 = s < 1$ leads to a change in level by $\Delta\beta = (10\text{dB})\log s$. Since $s < 1$ and $\log s < 0$, it follows that $\Delta\beta < 0$ dB.

Exr 5.2.35 If the sound level of a wave has decreased by 7.5 dB after traveling 31 cm, what is the attenuation coefficient measured in dB/cm?
We are told that the sound level has *decreased* by 7.5 dB. This means that $\Delta\beta = -7.5$ dB. From the definition $\alpha = -\Delta\beta/\Delta x$ we have that $\alpha = -(-7.5\text{ dB})/(31\text{ cm}) = 0.242$ dB/cm. Note that, unlike we would usually, we do *not* convert the distance to metres since the attenuation coefficient is measured per centimetre.

Exr 5.2.36 The sound level of a wave traveling through air decreases by 80 dB due to attenuation after traveling 333 m. What is the attenuation coefficient measured in dB/km?
Note that we were asked to find the attenuation coefficient measured in decibels per kilometre, so we will express the distance in those units. We have that $\alpha = -\Delta\beta/\Delta x = -(-80\text{ dB})/(0.333\text{ km}) = 240$ dB/km.

Exr 5.2.37 Ultrasound waves ($f = 4$ MHz) passing through muscle decrease by 13.7 dB after traveling 2.71 cm. What is the attenuation coefficient measured in dB/cm?
We have that $\alpha = -\Delta\beta/\Delta x = -(-13.7\text{ dB})/(2.71\text{ cm}) = 5.06$ dB/cm.

Exr 5.2.38 If a wave is propagating through a material with an attenuation coefficient of 5.4 dB/cm, by what amount will its sound level have decreased after propagating 3.3 cm?
From the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta\beta = -\alpha\,\Delta x = -(5.4\text{ dB/cm})(3.3\text{ cm}) = -17.8$ dB.

Exr 5.2.39 Ultrasonic waves of frequency 1 MHz propagates through air with an attenuation coefficient of 12.0 dB/cm. By what amount will the wave's sound level have decreased after propagating 2.20 mm?
Being careful to convert the distance to centimetres, from the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta\beta = -\alpha\,\Delta x = -(12.0\text{ dB/cm})(0.220\text{ cm}) = -2.64$ dB.

Exr 5.2.40 Ultrasonic waves of frequency 10 MHz propagates through water with an attenuation coefficient of 0.232 dB/cm. By what amount will the wave's sound level have decreased after propagating 43.2 cm?
Noting that the attenuation coefficient is given per metre, we must be careful to convert the distance to metres as well. From the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta\beta = -\alpha\,\Delta x = -(23.2\text{ dB/m})(0.432\text{ m}) = -10.0$ dB.

Exr 5.2.41 If a wave is propagating through a material with an attenuation coefficient of 8.0 dB/cm, what distance would it have to propagate before its sound level has decreased by 37 dB?
From the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta x = -\Delta\beta/\alpha = -(-37\text{ dB})/(8.0\text{ dB/cm}) = +4.63$ cm.

Exr 5.2.42 Ultrasound at 3 MHz propagates through muscle tissue with an attenuation coefficient of 4.15 dB/cm. What depth will it penetrate before its sound level has decreased by 22.0 dB?
From the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta x = -\Delta\beta/\alpha = -(-22.0\text{ dB})/(4.15\text{ dB/cm}) = +5.30$ cm.

Exr 5.2.43 High-frequency audible sound waves of 10 kHz propagates through air with an attenuation coefficient of 190 dB/km when the humidity is 40%. What distance would sound of that frequency have to travel before its sound level had decreased by 10.0 dB due to attenuation alone?
From the definition $\alpha = -\Delta\beta/\Delta x$ we get $\Delta x = -\Delta\beta/\alpha = -(-10.0\text{ dB})/(190\text{ dB/km}) = +0.0526\text{ km} = +52.6$ m.

5.2.5 Problems

PROBLEM 5.2.05: An ultrasound device is set to produce waves of frequency 3.00 MHz. If the attenuation of this wave is 3.8 dB/cm as it propagates into muscle tissue, at what depth (measured in millimetres) is 50% of the energy delivered?

If 50% of the energy has been delivered, then the intensity has dropped to 50% (half) of its initial value: $I_2 = \frac{1}{2} I_1$. The change in level is $\Delta\beta = (10\text{dB})\log(1/2) = -3.0103\,\text{dB}$.

From the definition $\alpha = -\Delta\beta/\Delta x$ the depth at which a given decrease occurs is given by $\Delta x = -\Delta\beta/\alpha$. The decrease in level (the $-3.0103\,\text{dB}$ found above). This happens over a distance

$\Delta x = -\Delta\beta/\alpha = -(-3.0103\,\text{dB})/(3.8\,\text{dB/cm}) = +0.7922\,\text{cm} = 7.92\,\text{mm}$.

PROBLEM 5.2.06: An ultrasound device is set to produce waves of frequency 1.00 MHz. If the attenuation of this wave is 4.95 dB/cm as it propagates through tendon, at what depth (measured in millimetres) is 37% of the energy delivered?

If 37% of the energy has been delivered, then the intensity has dropped to 63% of its initial value: $I_2 = 0.63\, I_1$. The change in level is $\Delta\beta = (10\text{dB})\log(0.63) = -2.007\,\text{dB}$.

From the definition $\alpha = -\Delta\beta/\Delta x$ the depth at which a given decrease occurs is given by $\Delta x = -\Delta\beta/\alpha$. The decrease in level (the $-3.0103\,\text{dB}$ found above). This happens over a distance

$\Delta x = -\Delta\beta/\alpha = -(-2.007\,\text{dB})/(4.95\,\text{dB/cm}) = +0.4054\,\text{cm} = 4.05\,\text{mm}$.

PROBLEM 5.2.07: In the human body sound waves travel at about 1540 m/s. An ultrasound device is set to produce waves of frequency 4.00 MHz.
(a) What is the wavelength (in millimetres) of these waves, in the body?
(b) If the attenuation of the wave is 8.2 dB/cm, at what depth (measured in millimetres) is 90% the energy delivered?
(c) What is this depth, expressed as a multiple of the wavelength?

(a) From the fundamental relation for periodic waves $\lambda = v/f = (1540\,\text{m/s})/(4.00\times10^{+6}\,\text{Hz}) = 3.85\times10^{-4}\,\text{m} = 0.385\,\text{mm}$.

(b) If 90% of the energy has been delivered, then the intensity has dropped to 10% (one-tenth) of its initial value: $I_2 = \frac{1}{10} I_1$. The change in level is $\Delta\beta = (10\text{dB})\log(1/10) = -10\,\text{dB}$.

From the definition $\alpha = -\Delta\beta/\Delta x$ the depth at which a given decrease occurs is given by $\Delta x = -\Delta\beta/\alpha$. The decrease in level (the $-10\,\text{dB}$ found above). This happens over a distance

$\Delta x = -\Delta\beta/\alpha = -(-10\,\text{dB})/(8.2\,\text{dB/cm}) = 1.22\,\text{cm} = 12.2\,\text{mm}$.

(c) The depth $\Delta x = 1.22\,\text{cm}$ is $\Delta x/\lambda = (12.2\,\text{mm})/(0.385\,\text{mm}) = 31.7$ wavelengths.

PROBLEM 5.2.08: The attenuation coefficient of skin at 2 MHz is approximately 5 dB/cm. If 2 MHz ultrasound of intensity 0.64 W/cm^2 is applied over an 12 cm-diameter circular area of skin that is 3 mm thick:
(a) What is the intensity of the wave just below the skin?
(b) What amount of energy is delivered into the portion below skin after seven minutes?

(a) Given the attenuation coefficient and the distance traveled, the change in sound level is $\Delta\beta = -\alpha\,\Delta x = -(5\,\text{dB/cm})(0.3\,\text{cm}) = -1.5\,\text{dB}$. From the relation $\Delta\beta = (10\text{dB})\log s$, the ratio of intensities ($s = I_1/I_2$) is equal to

$$(10\text{dB})\log s = \Delta\beta \tag{5.27}$$

$$s = 10^{\Delta\beta/(10\text{dB})} = 10^{(-1.5\text{dB})/(10\text{dB})} = 0.708 \tag{5.28}$$

Consequently the intensity below the skin is

$$I_2/I_1 = s \tag{5.29}$$

$$I_2 = s\times I_1 = (0.708)(0.64\,\text{W/cm}^2) = 0.45\,\text{W/cm}^2 \tag{5.30}$$

(b) With the intensity found in part (a), the energy delivered is

$$\Delta E = I_2\times A\times\Delta t = (0.45\,\text{W/cm}^2)\left(\pi(\tfrac{1}{2}\times 12\,\text{cm})^2\right)(7\times 60\,\text{s}) = 2.1375\times10^{+4}\,\text{J} = 21.4\,\text{kJ} \tag{5.31}$$

That is enough energy to raise the temperature of one kilogram of water by 5 C°.

Chapter 6

Electricity

"Electricity" is broad term for the phenomena related to the *controlled* transfer of electric charge. In the context of *electric circuits* this allows for the controlled transfer and transformation of *electric energy*.

6.1 Electric Current, Voltage and Power

Electric charge changing location transfers electric energy. Interaction of these charges with their surrounding material can then transform their electric energy into other forms. With current measuring the rate of charge motion it becomes meaningful to speak of the power transferred by current. The following exercises explore these ideas.

6.1.1 Current, Charge and Time

Electric ***charge*** is an intrinsic physical property (like mass) of the fundamental particles of matter.

Electric ***current*** measures the rate at which electric charge moves from place to place.

$$I = \frac{|\Delta Q|}{\Delta t} \tag{6.1}$$

With charge measured in coulombs and time in seconds, current is measured in *amperes* (1 A=1 C/s). Since a coulomb is a large amount of charge we will quite frequently be measuring currents in smaller increments such as milliamperes mA and microamperes µA. Notice that this quantity depends upon the *magnitude* of the charge, and not its sign.

Because of the relation between charge, time and current, an alternate unit for charge is the millampere-hour ($1\,\text{mAh} = 0.001\,\text{A} \times 3600\,\text{s} = 3.6\,\text{C}$). This unit is often used for measuring the capacity of small batteries, like those found in portable electronics.

Exr 6.1.01 What is the electric current (measured in milliamperes) when 75.0 C of charge is transferred in 120 s?
$I = |\Delta Q|/\Delta t = (75.0\,\text{C})/(120\,\text{s}) = 0.625\,\text{A} = 625\,\text{mA}$.

Exr 6.1.02 What is the electric current (measured in microamperes) when 632 µC of charge is transferred in 7.02 s?
$I = |\Delta Q|/\Delta t = (632 \times 10^{-6}\,\text{C})/(7.02\,\text{s}) = 90.0\,\mu\text{A}$.

Exr 6.1.03 What is the electric current (measured in amperes) when $37.0 \times 10^{+12}$ electrons are transferred in 1.00 s?
The amount of charge is $(37.0 \times 10^{+12}) \times (-1.602 \times 10^{-19}\,\text{C}) = -5.93\,\mu\text{C}$. This is a large number of electrons, but each has a very small charge, so the total is small.

The corresponding current is $I = |\Delta Q|/\Delta t = |(-5.93\,\mu\text{C})|/(1.00\,\text{s}) = 5.93\,\mu\text{A}$.

Exr 6.1.04 What is the electric current (measured in amperes) when one-tenth of a mole of electrons are transferred in one hour?
"One mole" of things is Avogadro's number of things $N_A = 6.022 \times 10^{+23}$. One-tenth of a mole of electrons is a charge of $(\frac{1}{10} \times 6.022 \times 10^{+23}) \times (-1.602 \times 10^{-19}\,\text{C}) = -9\,647\,\text{C}$. The charge of each electron is very small, but a mole is a profoundly large number, so the total charge is significant.

The corresponding current is $I = |\Delta Q|/\Delta t = |(-9\,647\,\text{C})|/(3\,600\,\text{s}) = 2.68\,\text{A}$. This sort of transfer might happen when operating an oven to cook a meal at medium temperatures, for example.

Exr 6.1.05 What is the electric charge (measured in coulombs) transferred when a current of 0.500 A flows for 12.0 s?
$|\Delta Q| = I \times \Delta t = (0.500\,\text{A}) \times (12.0\,\text{s}) = 6.00\,\text{C}$.

Exr 6.1.06 What is the electric charge (measured in coulombs) transferred when a current of 128 mA flows for seven and a half minutes?
The time elapsed is $\Delta t = 7.5 \times 60\,\text{s} = 450\,\text{s}$. The charge transferred is thus $|\Delta Q| = I \times \Delta t = (0.128\,\text{A}) \times (450\,\text{s}) = 57.6\,\text{C}$.

Exr 6.1.07 What is the electric charge (measured in milliamp-hours) transferred when a current of 88.8 mA flows for 20 minutes and 16 seconds?
The time elapsed is $\Delta t = 16\,\text{s} + (20 \times 60\,\text{s}) = 1216\,\text{s}$. The

charge transferred is thus $|\Delta Q| = I \times \Delta t = (0.0888\,\mathrm{A}) \times (1216\,\mathrm{s}) = 108\,\mathrm{C}$. This equals $108\,\mathrm{C} \times \frac{1\,\mathrm{mAh}}{3.6\,\mathrm{C}} = 30\,\mathrm{mAh}$.

Exr 6.1.08 If a current of 210 mA flows for 7 minutes how many moles of electrons are transferred?
The time elapsed is $\Delta t = 7 \times 60\,\mathrm{s} = 420\,\mathrm{s}$. The charge transferred is thus $|\Delta Q| = I \times \Delta t = (0.210\,\mathrm{A}) \times (420\,\mathrm{s}) = 88.2\,\mathrm{C}$. This equals $88.2\,\mathrm{C}/|-1.602 \times 10^{-19}\,\mathrm{C}| = 5.51 \times 10^{+20}$ electrons (where we used the magnitude of the charge because we are *counting*). This equals $(5.51 \times 10^{+20})/(6.022 \times 10^{+23}\mathrm{mol}^{-1}) = 9.14 \times 10^{-4}\mathrm{mol}$, a little less than one thousandth of a mole.

Exr 6.1.09 What amount of time (measured in minutes and seconds) must elapse for a 1.30 A current to transfer 420 C?
$\Delta t = |\Delta Q|/I = (420\,\mathrm{C})/(1.30\,\mathrm{A}) = 323\,\mathrm{s}$, which is five minutes and twenty-three seconds.

Exr 6.1.10 What amount of time (measured in hours and minutes) must elapse for a 0.750 A current to transfer 1200 mAh?
$\Delta t = |\Delta Q|/I = (1200\,\mathrm{mAh})/(750\,\mathrm{mA}) = 1.60\,\mathrm{h}$, which is one hour and thirty-six minutes. (Notice that our current must be measured in milliamperes when our charge is measured in millamp-hours)

Exr 6.1.11 Car batteries have capacities measured in amp-hours (not *milli*amp-hours). What amount of time (measured in minutes and seconds) must elapse for a 833 A current to drain 70 Ah from a car battery?
The amount of charge is $70\,\mathrm{Ah} = 70 \times (1\,\mathrm{A} \times 3600\,\mathrm{s}) = 2.52 \times 10^{+5}\,\mathrm{C} = 0.252\,\mathrm{MC}$. The time elapsed will thus be $\Delta t = |\Delta Q|/I = (0.252\,\mathrm{MC})/(833\,\mathrm{A}) = 303\,\mathrm{s}$, which is five minutes and three seconds.

6.1.2 Voltage, Charge and Energy

Similar to how there is gravitational energy between separated masses there is *electrical* energy between separated charges. The electrical potential difference, commonly referred to as the *voltage difference*, is defined by

$$\Delta V = \frac{\Delta E}{\Delta Q} \tag{6.2}$$

This is the ratio of the change in electrical energy to the amount of charge that experienced that change. (The analogous quantity in the case of gravity would be $(mg\,\Delta y)/m = g\,\Delta y$, which only depends upon the interaction and its geometry, but not the thing that is being interacted with.) The units of voltage difference are *volts*: $1\,\text{V} = 1\,\text{J/C}$.

As an example, if $-72\,\text{mC}$ charge is transferred across a $+1.50\,\text{V}$ difference, the change in electrical energy is $\Delta E = \Delta V \times \Delta Q = (+1.50\,\text{V}) \times (-0.072\,\text{C}) = -0.108\,\text{J}$. It is ***critical*** to note that, unlike the definition of current, the *signs* of the quantities involved are important.

Exr 6.1.12 What voltage difference was crossed by $+0.700\,\text{mC}$ if its electric energy changed by $+8.40\,\text{mJ}$?
$\Delta V = \Delta E/\Delta Q = (+8.40\,\text{mJ})/(+0.700\,\text{mC}) = +12.0\,\text{V}$.

Exr 6.1.13 What voltage difference was crossed by $+0.700\,\text{C}$ if its electric energy changed by $-77.7\,\text{mJ}$?
Remembering that signs are important, $\Delta V = \Delta E/\Delta Q = (-0.0777\,\text{J})/(+0.700\,\text{C}) = -0.111\,\text{V} = -111.\,\text{mV}$.

Exr 6.1.14 What voltage difference was crossed by an electron if its electric energy changed by $-0.2755 \times 10^{-15}\,\text{J}$?
The charge of an electron is $\Delta Q = -1.602 \times 10^{-19}\,\text{C}$. Thus $\Delta V = \Delta E/\Delta Q = (-0.2755 \times 10^{-15}\,\text{J})/(-1.602 \times 10^{-19}\,\text{C}) = +1720\,\text{V} = +1.720\,\text{kV}$.

Exr 6.1.15 What voltage difference was crossed by $+0.833\,\text{mAh}$ if its electric energy changed by $-27.0\,\text{J}$?
The charge is $\Delta Q = +0.833 \times 3.6\,\text{C} = +3.00\,\text{C}$. Thus $\Delta V = \Delta E/\Delta Q = (-27.0\,\text{J})/(+3.00\,\text{C}) = -9.00\,\text{V}$.

Exr 6.1.16 If $+2.11\,\text{mC}$ crosses a voltage difference of $-0.570\,\text{V}$ what is the change in the charge's electrical energy?
Remembering that the signs are important: $\Delta E = \Delta V \times \Delta Q = (-0.570\,\text{V}) \times (+2.11\,\text{mC}) = -1.20\,\text{mJ}$.

[By conservation of energy a decrease in the charge's electric energy would result in an equal increase in some other forms of energy in the system; either the charge's kinetic energy (like if it we accelerating towards a negatively charged object), or the thermal energy of its surroundings (if it were moving through a resistor in a circuit).]

Exr 6.1.17 If $-73.5\,\mu\text{C}$ crosses a voltage difference of $-4.44\,\text{mV}$ what is the change in the charge's electrical energy?
Remembering that the signs are important: $\Delta E = \Delta V \times \Delta Q = (-4.44\,\text{mV}) \times (-73.5\,\mu\text{C}) = +326.\,\text{nJ}$.

Exr 6.1.18 If $-25.6 \times 10^{-12}\,\text{C}$ crosses a voltage difference of $+5.08 \times 10^{+4}\,\text{V}$ what is the change in the charge's electrical energy?
$\Delta E = \Delta V \times \Delta Q = (+5.08 \times 10^{+4}\,\text{V}) \times (-25.6 \times 10^{-12}\,\text{C}) = -1.30\,\mu\text{J}$.

[This situation could be something like a large spark jumping a gap of about 1.6 cm. The decrease in the charge's electric energy manifests as an increase in the kinetic energy of the charges (as they accelerate across the gap), followed by dissipation by heat, light and sound.]

Exr 6.1.19 If $+295\,\text{mAh}$ crosses a voltage difference of $+12.0\,\text{V}$ what is the change in the charge's electrical energy?
The charge is $\Delta Q = +295 \times 3.6\,\text{C} = +1062\,\text{C}$. Thus $\Delta E = \Delta V \times \Delta Q = (+12.0\,\text{V}) \times (+1062\,\text{C}) = +12744\,\text{J} = 12.7\,\text{kJ}$.

[This situation would be like the energy delivered by a car battery during starting.]

Exr 6.1.20 What amount of charge needs to cross $640\,\text{mV}$ to transfer $10.24\,\mu\text{J}$?
$\Delta Q = \Delta E/\Delta V = (10.24\,\mu\text{J})/(640\,\text{mV}) = 16.0\,\mu\text{C}$.

Exr 6.1.21 What amount of charge (measured in mAh) needs to cross $1.500\,\text{V}$ to transfer $15.12\,\text{kJ}$?
the amount of charge is $\Delta Q = \Delta E/\Delta V = (15.12\,\text{kJ})/1.500\,\text{V} = 10080\,\text{C}$. Converting to milliamphours $\Delta Q = 10080\,\text{C} \times \frac{1\,\text{mAh}}{3.6\,\text{C}} = 2800\,\text{mAh}$. [This is the capacity of a standard AA-battery.]

6.1.3 Electrical Power

Charges moving across a voltage difference transfer energy. When current flows across a voltage difference, the rate of charge motion relates to the rate of energy transfer, which is *power*:

$$P = I \times \Delta V \tag{6.3}$$

Consider the units of current times voltage:

$$\mathrm{A} \times \mathrm{V} = \frac{\mathrm{C}}{\mathrm{s}} \times \frac{\mathrm{J}}{\mathrm{C}} = \frac{\mathrm{J}}{\mathrm{s}} = \mathrm{W} \tag{6.4}$$

Also, from the relation between power and energy, we have that

$$\Delta E = P \times \Delta t \tag{6.5}$$

$$= I \times \Delta V \times \Delta t \tag{6.6}$$

Exercise 6.1.22 What power is being delivered by 250 mA flowing through a 1.50 V AA-battery?
$P = I \times \Delta V = (0.250\,\mathrm{A}) \times (1.50\,\mathrm{V}) = 0.375\,\mathrm{W}$.

Exercise 6.1.23 What power is being delivered by 62 A flowing through a 12 V car battery?
$P = I \times \Delta V = (62\,\mathrm{A}) \times (12\,\mathrm{V}) = 744\,\mathrm{W}$ (which is almost one horse-power $1\,\mathrm{hp} = 746\,\mathrm{W}$).

Exercise 6.1.24 What power is being delivered by 2.15 A flowing through a resistor with a 5.52 V voltage difference across it?
$P = I \times \Delta V = (2.15\,\mathrm{A}) \times (5.52\,\mathrm{V}) = 11.9\,\mathrm{W}$.

Exercise 6.1.25 Charging a 9.00 V battery by forcing 0.987 A through it delivers what power?
$P = I \times \Delta V = (0.987\,\mathrm{A}) \times (9.00\,\mathrm{V}) = 8.88\,\mathrm{W}$.

Exercise 6.1.26 What energy is delivered by 325 mA flowing across 1.50 V after 30 s?
$\Delta E = I \times \Delta V \times \Delta t = (0.325\,\mathrm{A}) \times (1.50\,\mathrm{V}) \times (30\,\mathrm{s}) = 14.6\,\mathrm{J}$.

Exercise 6.1.27 What energy is delivered by 0.850 mA flowing across 33.3 mV after seventeen minutes?
$\Delta E = I \times \Delta V \times \Delta t = (8.50 \times 10^{-4}\,\mathrm{A}) \times (0.0333\,\mathrm{V}) \times (17 \times 60\,\mathrm{s}) = 28.9\,\mathrm{mJ}$.

Exercise 6.1.28 What current must flow across 120 V is required to deliver 1 500 W?
$I = P/\Delta V = (1\,500\,\mathrm{W})/(120\,\mathrm{V}) = 12.5\,\mathrm{A}$.

Exercise 6.1.29 What current must flow across 9.00 V is required to deliver 7.20 W?
$I = P/\Delta V = (7.20\,\mathrm{W})/(9.00\,\mathrm{V}) = 0.800\,\mathrm{A}$.

6.1.4 Problems

PROBLEM 6.1.01: A total of 720 mAh of electrons flows through a 1.500 V battery.
(a) What amount of charge in coulombs (including its sign) crossed the battery?
(b) If this charge *gained* electrical energy, what was the *sign* of the voltage difference?
(c) What was the change in the charge's electrical energy?
(d) If the battery was providing this energy at a constant rate of 300 mW, what was the current?

(a) The charge is composed of electrons, so it is negative.
Converting from milliamp-hours to coulombs: $\Delta Q = -720\,\text{mAh} = -720 \times 3.6\,\text{C} = -2\,592\,\text{C}$.

(b) If the charge gains energy, then $\Delta E > 0\,\text{J}$. Since $\Delta V = \Delta E/\Delta Q$ and $\Delta Q < 0\,\text{C}$, we must have $\Delta V < 0\,\text{V}$.

(c) From the definition of voltage difference $\Delta E = \Delta V \times \Delta Q = (-1.500\,\text{V}) \times (-2\,592\,\text{C}) = +3\,888\,\text{J}$.

(d) There are two ways to solve this problem.

Since power, voltage and current relate as $P = I\,\Delta V$, we see that $I = P/\Delta V = (0.300\,\text{W})/(1.500\,\text{V}) = 0.200\,\text{A} = 200\,\text{mA}$.

Alternatively, Since $P = \Delta E/\Delta t$, we have $\Delta t = \Delta E/P = (3\,888\,\text{J})/(0.300\,\text{W}) = 12\,960\,\text{s} = 3.60\,\text{h}$.
Thus $I = |\Delta Q|/\Delta t = (720\,\text{mAh})/(3.60\,\text{h}) = 200\,\text{mA}$.

PROBLEM 6.1.02: Desk-top cup warmer:
(a) How many AA batteries would you need to warm a cup of tea by $3.12\,\text{C}^\circ$? (Each 1.500 V battery can deliver 2 800 mAh. The cup contains 230 mL of tea, which has a heat capacity roughly that of water.)
(b) If we want to achieve this warming in five minutes, what must be the resistance of the heating element?

(a) The amount of water in the cup is $m = 230\,\text{mL} \times \frac{1\,\text{kg}}{1\,\text{L}} = 0.230\,\text{kg}$. The amount of thermal energy required to warm the cup is

$$\Delta E = m\mathscr{C}\,\Delta T = (0.230\,\text{kg})(4\,184\,\tfrac{\text{J}}{\text{kg}\cdot\text{C}^\circ})(+3.12\,\text{C}^\circ) = +3.00\,\text{kJ}$$

Each battery can deliver $\Delta E = \Delta Q \times \Delta V = (2\,800\,\text{mAh} \times \frac{3.6\,\text{C}}{1\,\text{mAh}}) \times (1.500\,\text{V}) = 15.1\,\text{kJ}$. One battery will do the job.

(b) Since $\Delta E = I \times \Delta V \times \Delta t$, the required current will be

$$I = \frac{\Delta E}{\Delta V \times \Delta t} = \frac{3.00\,\text{kJ}}{1.500\,\text{V} \times (5 \times 60\,\text{s})} = 6.67\,\text{A}$$

From Ohm's Law the resistance must be $R = \Delta V/I = (1.500\,\text{V})/(6.67\,\text{A}) = 0.225\,\Omega$.

6.2 Electric Circuits

Electric charge changing location (a *current*) transfers electric energy. Interaction of these charges with their surrounding material can then transform their electric energy into other forms, and vice versa. An ***electric circuit*** is an arrangement of circuit elements (like batteries, resistors, LEDs, etc) connected to each other by wires.

6.2.1 Ohm's Law & the Simple Circuit

Ohm's Law relates the difference in electrical potential (the voltage difference ΔV) across a resistor (R measured in ohms) to the current (I measured in amperes) that flows through it:

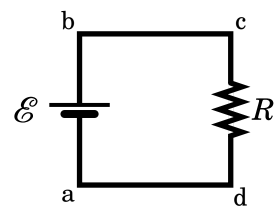
$$\Delta V = IR \tag{6.7}$$

The unit of resistance is the ohm: $1\,\text{ohm} = 1\,\Omega = 1\,\text{V/A}$. Since electrical power is given by $P = I\,\Delta V$, we also have

$$P = I^2 R \tag{6.8}$$

as the rate at which electric energy is dissipated as thermal energy by a resistor. This relation also shows that one ohm is equivalent to $1\,\Omega = 1\,\text{W/A}^2$. (Algebra can also show that $P = (\Delta V)^2/R$.)

The simplest circuit is a battery and a resistor connected in a single loop, as shown in the diagram below:



The voltage difference sustained by a battery (or generator, or other source of electrical energy) is referred to as the *electro-motive force*, or *emf* for short, denoted by the symbol $\mathscr{E}$. The value of the emf does not depend upon (or vary with) what it is connected to in the circuit. What does vary, and depends upon the other elements in the circuit, is the current that flows through the battery.

Exr 6.2.01 If the emf of the battery is $\mathscr{E} = 5.00\,\text{V}$ and the resistance in the circuit is $R = 220\,\Omega$, then find
(a) the current flowing around the circuit,
(b) the power dissipated by the resistor.

(a) Ohm's Law tells us that $I = \Delta V/R$. The voltage across the resistor is the emf of the battery, so $I = 5.00\,\text{V}/220\,\Omega = 0.0227\,\text{A} = 22.7\,\text{mA}$.

(b) The power dissipated by the resistor is $P = I^2R = (0.0227\,\text{A})^2(220\,\Omega) = 0.114\,\text{W} = 114\,\text{mW}$.

Exr 6.2.02 If the emf of the battery is $\mathscr{E} = 17.4\,\text{V}$ and the resistance in the circuit is $R = 421\,\Omega$, then find
(a) the current flowing around the circuit,
(b) the power dissipated by the resistor.

(a) Ohm's Law tells us that $I = \Delta V/R$. The voltage across the resistor is the emf of the battery, so $I = 17.4\,\text{V}/421\,\Omega = 0.0413\,\text{A} = 41.3\,\text{mA}$.

(b) The power dissipated by the resistor is $P = I^2R = (0.0413\,\text{A})^2(421\,\Omega) = 0.719\,\text{W} = 719\,\text{mW}$.

Exr 6.2.03 If the emf of the battery is $\mathscr{E} = 20.0\,\text{V}$ and the current flowing around the circuit is $I = 5.00\,\text{A}$, then find
(a) the value of the resistance,
(b) the power provided by the battery.

(a) Ohm's Law tells us that $R = \Delta V/I$. The voltage across the resistor is the emf of the battery, so $R = 20.0\,\text{V}/5.00\,\text{A} = 4.00\,\Omega$.

(b) Since the voltage across the battery is equal to its emf ($\Delta V = \mathscr{E}$) the power provided by the battery is $P = VI = \mathscr{E}I = 20.0\,\text{V} \times 5.00\,\text{A} = 100\,\text{W}$.

Exr 6.2.04 If the emf of the battery is $\mathscr{E} = 8.37\,\text{V}$ and the current flowing around the circuit is $I = 792\,\text{mA}$, then find
(a) the value of the resistance,
(b) the power provided by the battery.

(a) Ohm's Law tells us that $R = \Delta V/I$. The voltage across the resistor is the emf of the battery, so $R = 8.37\,\text{V}/0.792\,\text{A} = 10.6\,\Omega$.

(b) Since the voltage across the battery is equal to its emf ($\Delta V = \mathscr{E}$) the power provided by the battery is $P = VI = \mathscr{E}I = 8.37\,\text{V} \times 0.792\,\text{A} = 6.63\,\text{W}$.

Exr 6.2.05 If the emf of the battery is $\mathscr{E} = 8.00\,\mathrm{V}$ and it provides $P = 16.0\,\mathrm{W}$ of power, then find
(a) the current around the circuit,
(b) the value of the resistance.

(a) The power provided by the battery is $P = \mathscr{E}I$, so the current through the battery (and thus around the circuit) is $I = P/\mathscr{E} = (16.0\,\mathrm{W})/(8.00\,\mathrm{V}) = 2.00\,\mathrm{A}$.

(b) Ohm's Law tells us that $R = \Delta V/I$. The voltage across the resistor is the emf of the battery, so $R = 8.00\,\mathrm{V}/2.00\,\mathrm{A} = 4.00\,\Omega$.

Exr 6.2.06 If the emf of the power supply is $\mathscr{E} = 120\,\mathrm{V}$ and it provides $P = 1.200\,\mathrm{kW}$ of power, then find
(a) the current around the circuit,
(b) the value of the resistance.

(a) The power provided by the supply is $P = \mathscr{E}I$, so the current through the power supply (and thus around the circuit) is $I = P/\mathscr{E} = (1200\,\mathrm{W})/(120\,\mathrm{V}) = 10.0\,\mathrm{A}$.

(b) Ohm's Law tells us that $R = \Delta V/I$. The voltage across the resistance is the emf of the power supply, so $R = 120\,\mathrm{V}/10.0\,\mathrm{A} = 12.0\,\Omega$.

Exr 6.2.07 If the resistance is $R = 220\,\Omega$ and a current $I = 33.2\,\mathrm{mA}$ flows around the circuit, then find
(a) the emf of the battery,
(b) the power dissipated by the resistor.

(a) Ohm's Law gives us $\mathscr{E} = \Delta V = IR = (0.0332\,\mathrm{A})(220\,\Omega) = 7.30\,\mathrm{V}$.

(b) The power dissipated by a resistor is $P = I^2R = (0.0332\,\mathrm{A})^2(220\,\Omega) = 0.242\,\mathrm{W}$.

Exr 6.2.08 If the resistance is $R = 33.33\,\Omega$ and a current $I = 3.60\,\mathrm{A}$ flows around the circuit, then find
(a) the emf of the battery,
(b) the power dissipated by the resistor.

(a) the voltage across the resistor is the emf of the battery: $\Delta V = \mathscr{E}$. Thus Ohm's Law gives us $\mathscr{E} = IR = (3.60\,\mathrm{A})(33.33\,\Omega) = 120\,\mathrm{V}$. (This is the voltage of a wall plug.)

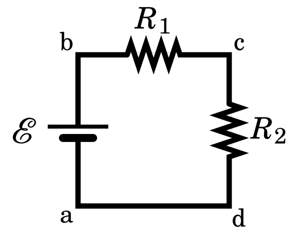
(b) The power dissipated by a resistor is $P = I^2R = (3.60\,\mathrm{A})^2(33.33\,\Omega) = 432\,\mathrm{W}$. (Let's hope this is designed as a heating element, else it is going to melt!)

6.2.2 Circuits with Resistors in Series

In *series*, the current through each element of the circuit (the battery, and each of the resistors) is the same. When resistors are connected in series the *equivalent resistance* of the resulting circuit is given by

$$R_{\text{eq}} = R_1 + R_2 \tag{6.9}$$

The current that flows through the battery, and each of the resistors, is given by $I = \mathcal{E}/R_{\text{eq}}$.



EXERCISE 6.2.09 If the emf of the battery in the series circuit is $\mathcal{E} = 5.00\,\text{V}$ and the resistances are $R_1 = 220\,\Omega$ and $R_2 = 150\,\Omega$, then find
(a) the current flowing around the circuit,
(b) the voltage difference across the small resistor,
(c) the power dissipated by the smaller resistor.

(a) The equivalent resistance of this series circuit is $R_{\text{eq}} = R_1 + R_2 = 220\,\Omega + 150\,\Omega = 370\,\Omega$. The current flowing out of the battery (and hence around the circuit) is found using Ohm's Law: $I = \mathcal{E}/R_{\text{eq}} = (5.00\,\text{V})/(370\,\Omega) = 0.0135135\,\text{A} = 13.5\,\text{mA}$.

(b) In series the current through each element of the circuit is the same. Ohm's Law thus gives the voltage across the smaller resistor: $V_2 = IR_2 = (0.0135135\,\text{A})(150\,\Omega) = 2.03\,\text{V}$.

(c) The power dissipated by the smaller resistor is $P_2 = I^2R_2 = (0.0135135\,\text{A})^2(150\,\Omega) = 0.0273923\,\text{W} = 27.4\,\text{mW}$.

EXERCISE 6.2.10 If the emf of the battery in the series circuit is $\mathcal{E} = 16.3\,\text{V}$ and the resistances are $R_1 = 53.0\,\Omega$ and $R_2 = 17.0\,\Omega$, then find
(a) the current flowing around the circuit,
(b) the voltage difference across the small resistor,
(c) the power dissipated by the smaller resistor.

(a) The equivalent resistance of this series circuit is $R_{\text{eq}} = R_1 + R_2 = 53.0\,\Omega + 17.0\,\Omega = 70.0\,\Omega$. The current flowing out of the battery (and hence around the circuit) is found using Ohm's Law: $I = \mathcal{E}/R_{\text{eq}} = (16.3\,\text{V})/(70.0\,\Omega) = 0.232857\,\text{A} = 233\,\text{mA}$.

(b) In series the current through each element of the circuit is the same. Ohm's Law thus gives the voltage across the smaller resistor: $V_2 = IR_2 = (0.232857\,\text{A})(17.0\,\Omega) = 3.96\,\text{V}$.

(c) The power dissipated by the smaller resistor is $P_2 = I^2R_2 = (0.232857\,\text{A})^2(17.0\,\Omega) = 0.921782\,\text{W} = 922\,\text{mW}$.

EXERCISE 6.2.11 If the emf of the battery in the series circuit is $\mathcal{E} = 70.0\,\text{V}$, the resistance $R_1 = 98.0\,\Omega$, and the current flowing around the circuit is $I = 345\,\text{mA}$, then find the value of the unknown resistor.

Given the emf and the current, the equivalent resistance of the circuit must be $R_{\text{eq}} = \mathcal{E}/I = (70.0\,\text{V})/(0.345\,\text{A}) = 203\,\Omega$. Because $R_{\text{eq}} = R_1 + R_2$, we find $R_2 = R_{\text{eq}} - R_1 = 203\,\Omega - 98.0\,\Omega = 105\,\Omega$.

EXERCISE 6.2.12 If the emf of the battery in the series circuit is $\mathcal{E} = 120\,\text{V}$, the resistance $R_1 = 34.3\,\Omega$, and the current flowing around the circuit is $I = 1.75\,\text{A}$, then find the value of the unknown resistor.

Given the emf and the current, the equivalent resistance of the circuit must be $R_{\text{eq}} = \mathcal{E}/I = (120\,\text{V})/(1.75\,\text{A}) = 68.6\,\Omega$. Because $R_{\text{eq}} = R_1 + R_2$, we find $R_2 = R_{\text{eq}} - R_1 = 68.6\,\Omega - 34.3\,\Omega = 34.3\,\Omega$.

EXERCISE 6.2.13 If the emf of the battery in the series circuit is $\mathcal{E} = 1.50\,\text{V}$, the resistance $R_1 = 1.22\,\Omega$, and the current flowing around the circuit is $I = 551\,\text{mA}$, then find the value of the unknown resistor.

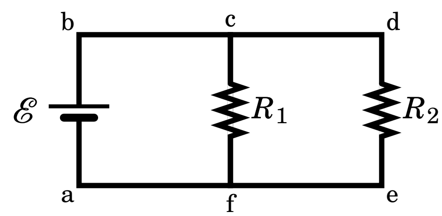
Given the emf and the current, the equivalent resistance of the circuit must be $R_{\text{eq}} = \mathcal{E}/I = (1.50\,\text{V})/(0.551\,\text{A}) = 2.72\,\Omega$. Because $R_{\text{eq}} = R_1 + R_2$, we find $R_2 = R_{\text{eq}} - R_1 = 2.72\,\Omega - 1.22\,\Omega = 1.50\,\Omega$.

6.2.3 Circuits with Resistors in Parallel

In *parallel*, the voltage across each element in the circuit (the battery, and each of the resistors) is the same. When resistors are connected in parallel the equivalent resistance of the resulting circuit is given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6.10}$$

Your must remember to take the reciprocal of this!: $R_{\text{eq}} = (1/R_{\text{eq}})^{-1}$. The current that flows through the battery is given by $I = \mathscr{E}/R_{\text{eq}}$.



EXERCISE 6.2.14 If the emf of the battery in the parallel circuit is $\mathscr{E} = 25.0\,\text{V}$ and the resistances are $R_1 = 11.0\,\Omega$ and $R_2 = 22.0\,\Omega$, then find
(a) the equivalent resistance of the circuit,
(b) the current out of the battery,
(c) the current through each resistor.

(a) The equivalent resistance of this parallel circuit is found by first calculating $(R_{\text{eq}})^{-1} = (R_1)^{-1} + (R_2)^{-1} = (11.0\,\Omega)^{-1} + (22.0\,\Omega)^{-1} = 0.13636\,\Omega^{-1}$. (Note carefully the units here, warning us that we are not yet done!) The equivalent resistance is thus $R_{\text{eq}} = (0.13636\,\Omega^{-1})^{-1} = 7.33\,\Omega$. (To evaluate this, note that it is less than either of the individual resistances, as it should be.)

(b) The current out of the battery is $I = \mathscr{E}/R_{\text{eq}} = (25.0\,\text{V})/(7.33\,\Omega) = 3.41\,\text{A}$.

(c) Remembering that the voltage across each element is the same in parallel ($V_1 = V_2 = \mathscr{E}$), the current through each resistor obeys Ohm's Law: $I_1 = \mathscr{E}/R_1 = (25.0\,\text{V})/(11.0\,\Omega) = 2.27\,\text{A}$; $I_2 = \mathscr{E}/R_2 = (25.0\,\text{V})/(22.0\,\Omega) = 1.14\,\text{A}$.

EXERCISE 6.2.15 If the emf of the battery in the parallel circuit is $\mathscr{E} = 75.0\,\text{V}$ and the resistances are $R_1 = 150\,\Omega$ and $R_2 = 68.0\,\Omega$, then find
(a) the equivalent resistance of the circuit,
(b) the current out of the battery,
(c) the current through each resistor.

(a) The equivalent resistance of this parallel circuit is found by first calculating $(R_{\text{eq}})^{-1} = (R_1)^{-1} + (R_2)^{-1} = (150\,\Omega)^{-1} + (68.0\,\Omega)^{-1} = 0.0213725\,\Omega^{-1}$, then calculating $R_{\text{eq}} = (0.0213725\,\Omega^{-1})^{-1} = 46.8\,\Omega$.

(b) The current out of the battery is $I = \mathscr{E}/R_{\text{eq}} = (75.0\,\text{V})/(46.8\,\Omega) = 1.60\,\text{A}$.

(c) Remembering that the voltage across each element is the same in parallel ($V_1 = V_2 = \mathscr{E}$), the current through each resistor obeys Ohm's Law: $I_1 = \mathscr{E}/R_1 = (75.0\,\text{V})/(150\,\Omega) = 0.500\,\text{A}$; $I_2 = \mathscr{E}/R_2 = (75.0\,\text{V})/(68.0\,\Omega) = 1.10\,\text{A}$.

EXERCISE 6.2.16 If the emf of the battery in the parallel circuit is $\mathscr{E} = 1.50\,\text{V}$ and the resistances are $R_1 = 0.375\,\Omega$ and $R_2 = 0.920\,\Omega$, then find
(a) the equivalent resistance of the circuit,
(b) the current out of the battery,
(c) the current through each resistor.

(a) The equivalent resistance of this parallel circuit is found by first calculating $(R_{\text{eq}})^{-1} = (R_1)^{-1} + (R_2)^{-1} = (0.375\,\Omega)^{-1} + (0.920\,\Omega)^{-1} = 3.75362\,\Omega^{-1}$, then calculating $R_{\text{eq}} = (3.75362\,\Omega^{-1})^{-1} = 0.266\,\Omega$.

(b) The current out of the battery is $I = \mathscr{E}/R_{\text{eq}} = (1.50\,\text{V})/(0.266\,\Omega) = 5.63\,\text{A}$.

(c) Remembering that the voltage across each element is the same in parallel ($V_1 = V_2 = \mathscr{E}$), the current through each resistor obeys Ohm's Law: $I_1 = \mathscr{E}/R_1 = (1.50\,\text{V})/(0.375\,\Omega) = 4.00\,\text{A}$; $I_2 = \mathscr{E}/R_2 = (1.50\,\text{V})/(0.920\,\Omega) = 1.63\,\text{A}$.

6.3 Time-Varying Currents

When electric current flows charges move through each element of the circuit. In the context of your *Electrotherapy* course a portion of the patient's body is an element of the circuit. In the human body it is a mixture of charged molecules, both positive and negative, that move when electric current flows in the body. With the boundaries of cells, organs, and ultimately the skin, electric current flowing in the human body has nowhere to go. Thus a current that flows at a constant rate through the human body will separate the charged molecules – with the positively charged molecules migrating to one side and the negatively charged molecules migrating to the other.

One method to avoid separating the different charges, while still permitting current, is alternate the direction of current flow. (Think of this like when you rub your hands together to produce heat by friction, and you alternate the direction of your hand's motion.)

Another method is to periodically decrease the applied voltage to zero. This gives the partially separated charges time to move back together and recombine into its original neutral mixture. That process is called *relaxation*.

Although the current may vary with time, all the relations between current, voltage and resistance still apply, since they apply *at each instant in time*. The same is true of the relation between current, voltage and power. So these relations are true at each instant in time:

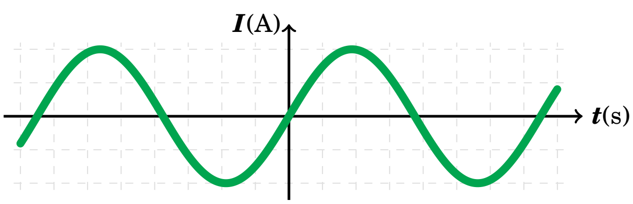
$$\Delta V = I \times R \tag{6.11}$$

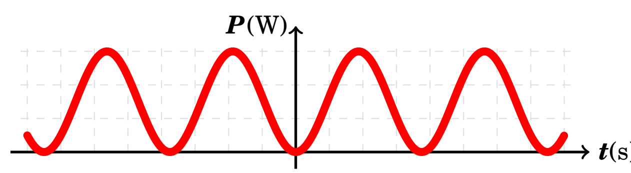
$$P = I \times \Delta V \tag{6.12}$$

What is complicated now are the relations between current and charge transported, and between power and energy transferred or transformed. These *totals* will now depend upon exactly how the current varies with time, and will often be expressed as *averages* over intervals of time that are large in comparison to the changes.

6.3.1 Alternating Current

One of the fundamental types of time-varying current is referred to as *Alternating Current*, which is abbreviated "AC". The current (and voltage) vary sinusoidally. (This is the type of current provided by electrical wall outlets.) If the current varies as the sine-function, then (as an example) the *instantaneous* power being dissipated by a resistor ($P = RI^2$) will vary as the square of the sine-function. This is graphed below:





The rate of energy dissipation does not depend upon the *direction* of charge motion, only its rate. So even as the current reverses direction – in the graph, where the current takes negative values over half of each cycle – the power is same as in the first half of the cycle. Calling the amplitude of the current oscillation I_{peak}, the maximum value of the power is $P_{\text{peak}} = R\,(I_{\text{peak}})^2$.

Averaged over many cycles the average power being dissipated is half of the peak value:

$$P_{\text{avg}} = \tfrac{1}{2} P_{\text{peak}} \tag{6.13}$$

This is defined by the fact that the total energy ΔE delivered by the alternating current will be $\Delta E = P_{\text{avg}}\,\Delta t$, where Δt is the total time that the current is oscillating. (This can be proved rigorously using calculus.)

This relation between the peak and average power is used to define a measure of the "average current". (Strictly speaking, since the current alternates direction, the current averages out to *zero*. The "average" being defined here is more like a measure of the *amplitude* of the oscillating current.) Defining I_{rms} the *root mean square* (RMS) average of the current through $P_{\text{avg}} = R\,(I_{\text{rms}})^2$, we obtain

$$P_{\text{avg}} = \tfrac{1}{2} P_{\text{peak}} \tag{6.14}$$

$$R\,(I_{\text{rms}})^2 = R\,(I_{\text{peak}})^2 \tag{6.15}$$

$$I_{\text{rms}} = \tfrac{1}{\sqrt{2}} I_{\text{peak}} \tag{6.16}$$

Since $\frac{1}{\sqrt{2}} \approx 0.7$ the RMS (average) value of the current is about 70% of the peak value. It is important to have these two measures of the alternating current because: (1) the average of the power relates to the target energy we seek to deliver; but (2) the peak current is important to control for safety reasons.

Through Ohm's Law the RMS (average) value of the oscillating voltage difference that drives the oscillating current is similarly defined in relation to its peak value:

$$\Delta V_{\text{rms}} = \frac{1}{\sqrt{2}} \Delta V_{\text{peak}} \tag{6.17}$$

(If you are aware of it, the "120 V" of the electrical supply from wall plugs is the RMS value of the oscillating voltage.) When oscillating voltages and currents are measured using "multimeters" it is usually the RMS values being reported. These relations hold:

$$\Delta V = I \times R \tag{6.18}$$

$$\Delta V_{\text{peak}} = I_{\text{peak}} \times R \tag{6.19}$$

$$\Delta V_{\text{rms}} = I_{\text{rms}} \times R \tag{6.20}$$

This first is true at each instant in time. The second relates the values at the instant the current is at its peak. The third relates the values when averaged over an interval of time that is long in comparison to the period of oscillation.

Frequency Dependence

The *expressions* for Ohm's Law ($\Delta V = I R$) and the instantaneous power ($P = I\,\Delta V$) do not depend upon the frequency of the current's alternation. What does vary with frequency, usually, is the resistance of the material, typically increasing with frequency – although the truth is quite complicated, and there are many exceptions. In practice, the frequency will usually be prescribed and the resistance under those conditions either known, or easily determined.

The resistance may vary with frequency, but the peak voltages and currents are independent of the frequency. Just remember that, as we saw for *waves*, amplitude and frequency are independent of each other.

Exr 6.3.01 If an alternating current $I_{\mathrm{rms}} = 13.0\,\mathrm{mA}$ flows through a $220\,\Omega$ resistor, find
(a) the RMS voltage being applied,
(b) the peak current flowing through the resistor, and
(c) the average power being dissipated.

(a) Ohm's Law tells us that $\Delta V_{\mathrm{rms}} = I_{\mathrm{rms}} \times R$. Thus $\Delta V_{\mathrm{rms}} = (0.0130\,\mathrm{A})(220\,\Omega) = 2.9\,\mathrm{V}$.

(b) The relation between peak and RMS current is $I_{\mathrm{rms}} = \frac{1}{\sqrt{2}} I_{\mathrm{peak}}$. The RMS value is about 70% of the peak, so the peak value should be bigger. Thus $I_{\mathrm{peak}} = \sqrt{2} \times I_{\mathrm{rms}} = \sqrt{2} \times 13.0\,\mathrm{mA} = 18.4\,\mathrm{mA}$.

(c) The average power relates to the RMS current by $P_{\mathrm{avg}} = R\,(I_{\mathrm{rms}})^2 = (220\,\Omega)(0.0130\,\mathrm{A})^2 = 37.2\,\mathrm{mW}$.

Exr 6.3.02 If an alternating current $I_{\mathrm{rms}} = 15.0\,\mathrm{A}$ flows through a $8.00\,\Omega$ resistor, find
(a) the RMS voltage being applied,
(b) the peak current flowing through the resistor, and
(c) the average power being dissipated.

(a) Ohm's Law tells us that $\Delta V_{\mathrm{rms}} = I_{\mathrm{rms}} \times R$. Thus $\Delta V_{\mathrm{rms}} = (15.0\,\mathrm{A})(8.00\,\Omega) = 120\,\mathrm{V}$. (This is plug voltage. Is this an appliance?)

(b) The relation between peak and RMS current is $I_{\mathrm{rms}} = \frac{1}{\sqrt{2}} I_{\mathrm{peak}}$. The RMS value is about 70% of the peak, so the peak value should be bigger. Thus $I_{\mathrm{peak}} = \sqrt{2} \times I_{\mathrm{rms}} = \sqrt{2} \times 15.0\,\mathrm{A} = 21.2\,\mathrm{A}$. (Definitely not a biologically safe current.)

(c) The average power relates to the RMS current by $P_{\mathrm{avg}} = R\,(I_{\mathrm{rms}})^2 = (8.00\,\Omega)(15.0\,\mathrm{A})^2 = 1.80\,\mathrm{kW}$. (Maybe an oven? Or a heater of some kind?)

Exr 6.3.03 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 120\,\mathrm{V}$ is applied across a $68.0\,\Omega$ resistor, find
(a) the RMS current flowing through the resistor,
(b) the peak current flowing through the resistor, and
(c) the average power being dissipated.

(a) Ohm's Law tells us that $I_{\mathrm{rms}} = \Delta V_{\mathrm{rms}}/R$. Thus $I_{\mathrm{rms}} = (120\,\mathrm{V})/(68.0\,\Omega) = 1.76\,\mathrm{A}$.

(b) The relation between peak and RMS current is $I_{\mathrm{rms}} = \frac{1}{\sqrt{2}} I_{\mathrm{peak}}$. The RMS value is about 70% of the peak, so the peak value should be bigger. Thus $I_{\mathrm{peak}} = \sqrt{2} \times I_{\mathrm{rms}} = \sqrt{2} \times 1.76\,\mathrm{A} = 2.49\,\mathrm{A}$.

(c) The average power relates to the RMS voltage by $P_{\mathrm{avg}} = (\Delta V_{\mathrm{rms}})^2/R = (120\,\mathrm{V})^2/(68.0\,\Omega) = 212\,\mathrm{W}$.

Exr 6.3.04 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 9.00\,\mathrm{V}$ is applied across a $725\,\mathrm{m}\Omega$ resistor, find
(a) the RMS current flowing through the resistor,
(b) the peak current flowing through the resistor, and
(c) the average power being dissipated.

(a) Ohm's Law tells us that $I_{\mathrm{rms}} = \Delta V_{\mathrm{rms}}/R$. Thus $I_{\mathrm{rms}} = (9.00\,\mathrm{V})/(0.725\,\Omega) = 12.4\,\mathrm{A}$.

(b) The relation between peak and RMS current is $I_{\mathrm{rms}} = \frac{1}{\sqrt{2}} I_{\mathrm{peak}}$. The RMS value is about 70% of the peak, so the peak value should be bigger. Thus $I_{\mathrm{peak}} = \sqrt{2} \times I_{\mathrm{rms}} = \sqrt{2} \times 12.4\,\mathrm{A} = 17.5\,\mathrm{A}$.

(c) The average power relates to the RMS voltage by $P_{\mathrm{avg}} = (\Delta V_{\mathrm{rms}})^2/R = (9.00\,\mathrm{V})^2/(0.725\,\Omega) = 112\,\mathrm{W}$.

Exr 6.3.05 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 75.0\,\mathrm{V}$ dissipates $210\,\mathrm{W}$ (average) in a resistor, find
(a) the RMS current flowing through the resistor, and
(b) the value of the resistance.

(a) The average power defines $P_{\mathrm{avg}} = (I_{\mathrm{rms}})^2 \times R$. By Ohm's Law ($\Delta V_{\mathrm{rms}} = I_{\mathrm{rms}} \times R$) this means that $P_{\mathrm{avg}} = I_{\mathrm{rms}} \times \Delta V_{\mathrm{rms}}$. Thus $I_{\mathrm{rms}} = P_{\mathrm{avg}}/\Delta V_{\mathrm{rms}} = (210\,\mathrm{W})/(75.0\,\mathrm{V}) = 2.80\,\mathrm{A}$.

(b) By Ohm's Law $R = \Delta V_{\mathrm{rms}}/I_{\mathrm{rms}} = (75.0\,\mathrm{V})/(2.80\,\mathrm{A}) = 26.8\,\Omega$.

Exr 6.3.06 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 120\,\mathrm{V}$ dissipates $15.3\,\mathrm{W}$ (average) in a resistor, find
(a) the RMS current flowing through the resistor, and
(b) the value of the resistance.

(a) The average power defines $P_{\mathrm{avg}} = (I_{\mathrm{rms}})^2 \times R$. By Ohm's Law ($\Delta V_{\mathrm{rms}} = I_{\mathrm{rms}} \times R$) this means that $P_{\mathrm{avg}} = I_{\mathrm{rms}} \times \Delta V_{\mathrm{rms}}$. Thus $I_{\mathrm{rms}} = P_{\mathrm{avg}}/\Delta V_{\mathrm{rms}} = (15.3\,\mathrm{W})/(120\,\mathrm{V}) = 128\,\mathrm{mA}$.

(b) By Ohm's Law $R = \Delta V_{\mathrm{rms}}/I_{\mathrm{rms}} = (120\,\mathrm{V})/(0.128\,\mathrm{A}) = 938\,\Omega$.

Exr 6.3.07 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 120\,\mathrm{V}$ is applied across a $22.2\,\Omega$ resistor, find the time required to dissipate $7.31\,\mathrm{kJ}$.

The average power relates to the energy delivered and time taken by $\Delta E = P_{\mathrm{avg}} \times \Delta t$, so that $\Delta t = \Delta E/P_{\mathrm{avg}}$. The average power is $P_{\mathrm{avg}} = (\Delta V_{\mathrm{rms}})^2/R = (120\,\mathrm{V})^2/(22.2\,\Omega) = 649\,\mathrm{W}$. Thus $\Delta t = \Delta E/P_{\mathrm{avg}} = (7\,310\,\mathrm{J})/(649\,\mathrm{W}) = 11.3\,\mathrm{s}$.

Exr 6.3.08 If an alternating voltage difference $\Delta V_{\mathrm{rms}} = 5.00\,\mathrm{V}$ is applied across a $417\,\mathrm{m}\Omega$ resistor, find the time required to dissipate $29.2\,\mathrm{kJ}$.

The average power relates to the energy delivered and time taken by $\Delta E = P_{\mathrm{avg}} \times \Delta t$, so that $\Delta t = \Delta E/P_{\mathrm{avg}}$. The average power is $P_{\mathrm{avg}} = (\Delta V_{\mathrm{rms}})^2/R = (5.00\,\mathrm{V})^2/(0.417\,\Omega) = 60.0\,\mathrm{W}$. Thus $\Delta t = \Delta E/P_{\mathrm{avg}} = (29\,200\,\mathrm{J})/(60.0\,\mathrm{W}) = 487\,\mathrm{s} \approx 8\,\mathrm{minutes}$.